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A NOTE ON EXACT PARTICULAR SOLUTIONS OF GENERALIZED SHALLOW-WATER EQUATIONS

By

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Abstract. This note presents a set of systems of two first-order quasi-linear partial differential equations, which can be reduced to the shallow-water equations. This set includes equations describing a two-layered fluid flow.

Introduction. The shallow-water equations

$$u_t+uu_x+h_x=0, \quad h_t+uh_x+hu_x=0 \tag{SW}$$

have been intensively studied analytically and numerically (see e.g. [2], [4, § 13.10]). Several exact analytical solutions of (SW) are known

$$u=x/t,\ h=a/t; \quad u=(b+2x/t)/3,\ h=(x/t-b)^2/9+a/t^{2/3}; \qquad [2, \S\ 5] \tag{PSa,b}$$

$$u=f'(t)x/f(t),\ h=0.25a(1-x^2/f^2(t))/f(t),\ f'(t)=(1-a/f(t))^{1/2}; \tag{PSc}$$

$$x=2ht^2s+0.5\ln(1+2s/(1-s)),\ u=2hts,\ s^2=1-h/(b-h^2t^2); \quad [3, \S\ 3]. \tag{PSd}$$

Here a and b are an arbitrary constants, ′ denotes the derivative.

Consider the generalized shallow-water equations

$$v_t+F(v,\zeta)v_x+0.5G_1(v,\zeta)\zeta_x=0, \quad \zeta_t+F(v,\zeta)\zeta_x+0.5G_2(v,\zeta)v_x=0. \tag{GSW}$$

Theorem. Let $u=u(x,t)$ and $h=h(x.t)$ be a solution of (SW).

a) If

$$F(v,\zeta)=f_1(v+\zeta)+f_2(v-\zeta) \text{ and } G_1(v,\zeta)=G_2(v,\zeta)=f_1(v+\zeta)-f_2(v-\zeta), \qquad (1)$$

then relations

$$f_1(v+\zeta)+f_2(v-\zeta)=u(x,t) \text{ and } f_1(v+\zeta)-f_2(v-\zeta)=2h^{1/2}(x,t) \qquad (2)$$

give a solution of (GSW).

b) If

$$F(v,\zeta)=f_1(v)f_2(\zeta)+q,\ G_1(v,\zeta)=(c_1+f_1^2)f_2'/f_1' \text{ and } G_2(v,\zeta)=(c_2+f_2^2)f_1'/f_2', \qquad (3)$$

then relations

$$f_1(v)f_2(\zeta)+q=u(x,t) \text{ and } (c_1+f_1^2)(c_2+f_2^2)=4h(x,t) \qquad (4)$$

give a solution of (GSW). Here f_1 and f_2 are arbitrary functions, q is a constant.

Proof. Inserting $u=F(v,\zeta)$ and $h=G(v,\zeta)$ into (SW), we get

$$v_t+(F+A(v,\zeta))v_x+B_1(v,\zeta)\zeta_x=0,\ \zeta_t+(F-A(v,\zeta))\zeta_x+B_2(v,\zeta)v_x=0. \qquad (5)$$

Here $A=(G_vG_\zeta-F_vF_\zeta)/D$, $B_1=(G_\zeta^{\,2}-GF_\zeta^{\,2})/D$, $B_2=(GF_v^{\,2}-G_v^{\,2})/D$ and $D(v,\zeta)=F_vG_\zeta-G_vF_\zeta$.

a) Considering $F(v,\zeta)=f_1(v+\zeta)+f_2(v-\zeta)$ and $G=[f_1(v+\zeta)-f_2(v-\zeta)]^2/4$ we obtain $A=0$, $B_1=B_2=G_1(v,\zeta)/2$. Thus (5) becomes (GSW) with F, G_1 and G_2 as given by (1).

b) Considering $F(v,\zeta)=f_1(v)f_2(\zeta)+q$ and $G=(c_1+f_1^2)(c_2+f_2^2)/4$ we obtain $A=0$, $B_1=0.5(c_1+f_1^2)f_2'/f_1'$, $B_2=0.5(c_2+f_2^2)f_1'/f_2'$. Thus (5) becomes (GSW) with F, G_1 and G_2 as given by (3).

Example 1. Taking $f_1(z)=f_2(z)=z^n$ in (1) we get (GSW) with $F=(v+\zeta)^n+(v-\zeta)^n$, $G=G=(v+\zeta)^n-(v-\zeta)^n$. This equation has exact analytical solutions $2v(x,t)=(u/2+h^{1/2})^{1/n}+(u/2-h^{1/2})^{1/n}$, $2\zeta(x,t)=(u/2+h^{1/2})^{1/n}-(u/2-h^{1/2})^{1/n}$ with u(x,t) and h(x,t) from (PSa-d).

Example 2. Unsteady two-layer fluid flow. Taking $f_1(x)=f_2(x)=x$ and $c_1=c_2=-1$ in (3) we get (GSW) in the form

$$v_t+(v\zeta+q)v_x-0.5(1-v^2)\zeta_x=0, \quad \zeta_t+(v\zeta+q)\zeta_x-0.5(1-\zeta^2)v_x=0. \qquad \text{(SW2)}$$

This system describes a flow of two homogenous inviscid fluids between two horizontal rigid plates in the hydrostatic and Boussinesq approximations (see (3.3.12) in [1]). The velocity u_1, u_2 and the thickness h_1, h_2 of the upper lighter (density ρ_1) and lower denser (density ρ_2) layer (see Fig.1) are expressed in terms of v and ζ as follows:

$$u_1/c_0=q+v(1+\zeta)/2,\ u_2/c_0=q-v(1-\zeta)/2,\ h_1/H=(1-\zeta)/2,\ h_2/H=(1+\zeta)/2.$$

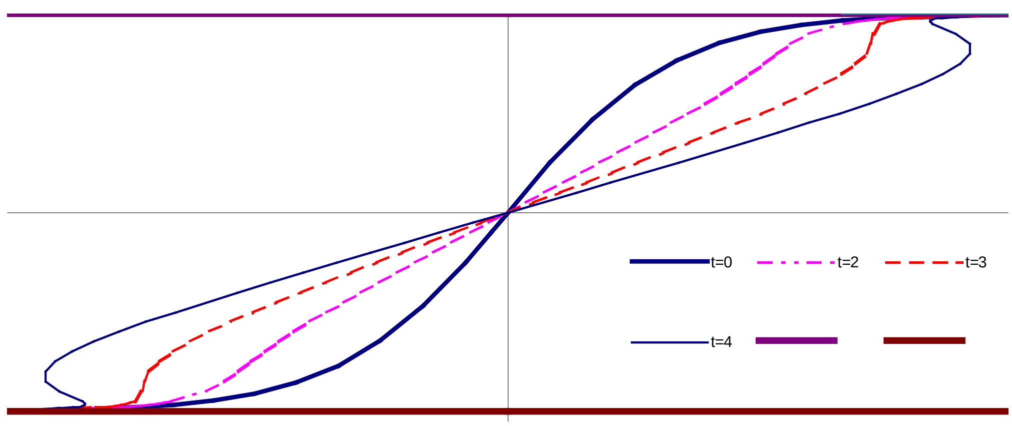

Fig.1 Evolution of the interface described by (SW2) with q=0 for initial conditions $v(x,0)=0$, $\zeta(x,0)=\tanh x$.

Here H is the distance between the plates, $c_0^2=g'H$, $g'=g(\rho_2-\rho_1)/\rho_2$, g is the acceleration of gravity, $q=Q/c_0H$, Q is the total volume flux, $-1<\zeta<1$.

If $u(x,t)$, $h(x,t)$ is a solution of (SW) then $v=w_+ + w_-$, $\zeta = w_+ - w_-$ is a solution of (SW2). Here $w_\pm=[(1\pm u(x-qt,t))^2/4-h(x-qt,t)]^{1/2}$. Using (PSa-d) we get exact analytical solutions of (SW2). In particular, using (PSd) with $b=1/4$ we have

$$x-qt=\zeta(1-\zeta^2)t^2/[1+(1+(1-\zeta^2)^2 t^2)^{1/2}]+0.5\ln(1+2\zeta/(1-\zeta)),$$

$$v=(1-\zeta^2)t/[1+(1+(1-\zeta^2)^2 t^2)^{1/2}].$$

This is the solution of (SW2) with initial conditions $\zeta(x,0)=\tanh x$, $v(x,0)=0$. Let $q=0$. For $x \ll 1$ we have $\zeta\approx x/(1+t^2)^{1/2}$ and $v\approx t/[1+(1+t^2)^{1/2}]$. The maximum of $v(x,t)$ is at $x=0$ and increases from 0 to 1 when t increases from 0 to infinity.

Fig. 1 shows the position of the interface for t=0, 2, 3 and 4.

References

[1] P.G. Baines, *Topographic effects in stratified flows*. Cambridge University Press, NY, (1995)

[2] J. Gratton and C. Vigo, *Self-similar gravity currents with variable inflow revisited: plane currents*, J. Fluid Mech. **258**, 77-104 (1994)

[3] A.B. Odulo, *A method of obtaining exact particular solutions of a system of two first-order quasi-linear partial differential equations*, Q. J. Mech. Appl. Math. **44(2)**, 323-331 (1992)

[4] G.B. Whitham, *Linear and nonlinear waves*, Wiley-Interscience, NY, (1974)