THE STEADY FLOW BETWEEN RESERVOIRS WITH DIFFERENT DENSITY AND LEVEL THROUGH A CONTRACTION

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ABSTRACT

This paper presents a complete analytical solution of steady gravity flow between two reservoirs connected by a channel of slowly varying breadth and containing fluids of different densities and levels. The hydrostatic approximation is used and dissipation is neglected.

It is shown that seven different regimes are possible depending on the value of the parameter $\delta=\gamma/\varepsilon$, which is the ratio of relative lighter and denser reservoir level difference, γ, to positive relative density difference, ε. The exact solution of the problem is obtained for all these regimes. If the level of the heavier fluid reservoir is higher than the level of lighter fluid reservoir, $\delta \leq 0$, then the denser fluid plunges under the lighter motionless fluid. If $\delta \geq 1$ the lighter fluid runs up on a wedge of the motionless denser fluid.

If $0<\delta<1$ two-directional exchange flow occurs. The exact analytical expressions for layer discharges for the entire range of the parameters ε and δ are found and discussed. Wood's (1970) experimental data with non small ε are in good agreement with the theory. When $\varepsilon \to 0$ an exchange regime exists as long as $\gamma \to 0$ to keep their ratio between 0 and 1, $1>\gamma/\varepsilon>0$. At this limit the existence of an exchange flow and the solution depend only on the ratio γ/ε, not the values of γ and ε individually, and the Boussinesq approximation can be used.

Some examples of application of the theory to prediction of mass and volume transport through a contraction for steady and quasi-steady flows (due to slowly vary in the basins level) are given.

1. INTRODUCTION

The gravitational flow of two fluids of different densities through a contraction is important in numerous engineering and geophysical problems (e.g. Schijf and Schönfeld (1953), Stommel and Farmer (1953), Bryden and Kinder (1991), Baines (1995), pp. 146-7, Hogg and Huang (1995), Chapter 4). In fact, observations of two-directional flow go back at least to the sixth century, when "the fishermen of the towns on the Bosphorus say that the whole stream does not flow in the direction of Byzantium, but while the upper current which we can see plainly does flow in this direction, the deep water of the abyss, as it is called, moves in a direction exactly opposite to that of the upper current and so flow continuously against the current which is seen" (Gill, 1977, p.96). In the seventeenth century Marsigli experimentally demonstrated that the density difference between the Black Sea and the Mediterranean causes a two-directional flow in the Strait. "He had attempted to measure a difference in sea level between the Black Sea and the Mediterranean using a mercury barometer" (Gill, 1977, p. 97). But if the density difference between the reservoirs is very small, the exchange flow exists only for a very small difference in sea level. To study the influence of the difference in sea level on two-directional flow Wood (1970) conducted an experiment with exchange flow between two reservoirs connected by a contraction, containing fluids of different densities (ρ_1 and $\rho_2>\rho_1$) and covered a third layer of lightest ($\rho_0<\rho_1$) stagnant fluid. In this case neither $\varepsilon=(\rho_2-\rho_1)/(\rho_2-\rho_0)$ was very small nor was the relative difference in reservoir levels, $\gamma=(H_1-H_2)/H$. The measurements of the thickness of the upper and lower layers (η_1 and η_2) were easily recorded photographically.

Wood (1970) showed that the condition that the thicknesses of the moving layers decrease smoothly from their values in the upstream reservoirs (H_1 and H_2) to the value of zero in the infinitely wide downstream reservoirs, gives two additional equations which provide the complete system of algebraic equations determining a unique solution. He presented the results of numerical calculations as graphs of η_1 and η_2 at the minimum width b_0 and q_1^2 and q_2^2 as functions of the depths ratio H_2/H_1 for a range of density difference ratios ε. Here $q=Q/(cb_0H)$ is a discharge coefficient, $c=\sqrt{2\varepsilon gH}$, b_0 is the channel width in the narrowest cross section, g is the gravitational acceleration; see also notation in Appendix E. He also obtained the solution for the case of a denser layer plunging under a stationary lighter layer.

Exchange flow in the case of the absence of the third layer ($\rho_0=0$) was considered by Armi and Farmer

(1986), Lawrence (1990), Dalziel (1991) (and in this paper); see also Baines (1995, §3.11). Changing notation one can make both problems (ρ_0=0 and the problem solved by Wood (1970)) identical. Armi and Farmer (1986) solved the problem numerically in the Boussinesq approximation using "net discharge", $U=q_1-q_2$, as the independent parameter. They also discussed the case of a denser layer plunging under a stationary lighter layer and the case of a lighter layer running up on a stationary denser layer. In the Boussinesq approximation Lawrence (1990) found algebraic expressions for the thicknesses of the layers in the narrowest section and algebraic expressions for values of channel width where the layer velocities are equal and where their sum is equal to $c/\sqrt{2}$. He used the discharge ratio $q_r=q_1/q_2$ as the independent parameter. For the case of pure contraction Dalziel (1991) presented numerical results identical to those obtained by Armi and Farmer (1986).

Most previous studies concentrated on the positions and flow conditions at so-called "control points". The main goal of our study is the determination of the discharges Q_1 and Q_2 in terms of external conditions (channel geometry, reservoir levels and fluid densities).

In this paper we obtain the exact analytical solution of the steady flow through a contraction which connects two large basins with fluids of given densities and levels. Our study is based on the specific energy equation, introduced by Bakhmeteff (1932, §15), see also Henderson (1966, p. 31). It will be shown that the key parameter of this problem is $\delta=(H_1-H_2)/\varepsilon H$, $H=\max(H_1,H_2)$. If $\delta < 0$ then the denser fluid plunges under the stagnant lighter layer (regimes 1-3), if $\delta > 1$ then the lighter fluid runs up on a wedge of stagnant heavier fluid (regimes 5-7). The results for these regimes are presented in section 2 and the solutions are given in Appendixes A and B. Regimes 1, 2, 3, 5, 6 and 7 differ in the positions of the tip of the wedge of the stagnant fluid (the plunge point, Wood, 1970, p. 676) and in the dependence of Q_1 and Q_2 on δ. For regimes 1 and 7 the discharges Q_1 and Q_2 are constants independent of ε and δ. The discharge $Q_1 \sim (\delta-1)^{1/2}$ for regime 6 and $Q_2 \sim (-\delta)^{1/2}$ for regime 2, $Q_1 \sim \delta^{3/2}$ for regime 5 and $Q_2 \sim (1-\delta)^{3/2}$ for regime 3.

If $0 < \delta < 1$ then both layers are in motion (regime 4). The complete analytical solution for the entire range of the parameter ε ($0 < \varepsilon < 1$) is given in section 3. In particular the exact expressions for the contraction discharge coefficients q_1 and q_2 as functions of the parameters ε and δ are found. For exchange flow graphs $q_1(\varepsilon,\delta)$ and $q_2(\varepsilon,\delta)$ are close to $q_1(0,\delta)$ and $q_2(0,\delta)$, respectively. In the Boussinesq approximation (ε=0; subsection 3.b.) the solution takes the simpler form. In particular, the discharge of lighter fluid $Q_1=A(\delta)\delta c b_0 H$ and the discharge of denser fluid $Q_2=A(\delta)(1-\delta)c b_0 H$ (here $A(\delta)$ varies between $2/\sqrt{27}$ (for δ=0 and δ=1) and $1/\sqrt{8}$ (for δ=0.5)).

The application of the steady solution in the case when δ and/or ε slowly change with time is discussed in section 4.

Let us now show the physical meaning of the parameter δ. If the reservoirs have the same densities, but different depths and are separated by a gate, the difference of their potential energies per unit mass is $g(H_1-H_2)/2$. One can find the description of the flow initiating from the removal of the gate for a channel of constant width in Henderson (1966, pp. 309-310). If the reservoirs have the same depths, but different densities and are separated by a gate, the difference of their potential energies per unit mass is $\varepsilon gH/2$. One can think of δ as the ratio of these differences of the potential energies.

Now we shall give the example of the initial condition which results in the steady flow considered in this paper. Let the reservoirs have different densities (ρ_1 and $\rho_2>\rho_1$), different depths (H_1 and H_2) and be separated by a gate in the narrowest cross-section. Then the pressure on the left side of the gate is $p_1= g\rho_1 z$ (here z is the vertical downward coordinate, $0<z<H_1$) and on the right side is $p_2=g\rho_2(z+H_2-H_1)$ (here $H_1-H_2<z<H_1$). If $H_2>H_1$ (this means $\delta < 0$) then $p_2>p_1$ at all z and after the removal of the gate the flow initially will be from right to left at all z. If $\rho_1 H_1>\rho_2 H_2$ (this means $\delta > 1$) then $p_1>p_2$ for all z and after the removal of the gate the flow initially will be from left to right for all z. If $H_1>H_2>\rho_1 H_1/\rho_2$ (this means $0< \delta < 1$) then $p_1>p_2$ for $0<z<\delta H_1$ and $p_2>p_1$ for $\delta H_1<z<H_1$, therefore after the removal of the gate the flow initially will be from left to right for $0<z<\delta H_1$ and from right to left for $\delta H_1<z<H_1$.

It is clear that the unsteady flows initiated by the removal of the gate in the cases where $H_1 \neq H_2$ and where the densities on the either side of the gates are equal in one case but different in the other, are similar only if $|\delta$

$|$>>1. The flows are very different if $|\delta|$ is order of or less than 1. In this paper we study the steady flow of two fluids with different densities.

The conventions adopted throughout this paper are as follows:
the lighter fluid moves from left to right and the denser fluid moves from right to left;
all values and parameters are positive except δ and γ which are negative when $H_1 < H_2$;
x is the horizontal coordinate along a channel;
the channel width b(x) is infinite at $x \to \pm\infty$ and has a unique minimum b_0 at x=0;
the channel has a flat bottom, $H=\max(H_1,H_2)$.

2. PLUNGING AND RUN UP

We consider the problem of two-layer flow through a rectangular profile channel of slowly varying width connecting two reservoirs of infinite width (plan view is sketched in Fig. 1(a); Fig. 1(b)-(h) present calculated side views for particular values of the parameters ε and δ and illustrate all seven regimes).

a. A denser fluid plunging under a stationary layer. If the level of the heavier fluid reservoir is higher than the level of lighter fluid reservoir ($H_1 < H_2$), then the lighter fluid is at rest (velocity $u_1=0$). The position of the plunge point (defined as the position of the tip of the wedge of motionless fluid) is denoted as x_*.

The solution can be found from the Bernoulli and continuity equations and the requirement that the thickness of the denser layer η_2 continuously decrease from H_2 to 0 (see Appendix A). The first three rows of Table 1 present the contraction discharge coefficients $q_1=Q_1/cHb_0$ and $q_2=Q_2/cHb_0$ and the non-dimensional thicknesses of the lighter and denser layers $\xi_1=\eta_1/H$ and $\xi_2=\eta_2/H$ at the position of minimum width at x=0 (which we denote as ξ_{10} and ξ_{20}, correspondingly) as functions of ε and δ for regimes 1-3. Regimes 1-3 (see Fig. 1(b)-(d)) correspond the position of the plunge point x_* being downstream, at and upstream of the narrowest section (Wood, 1970, p.676).

For given ε and δ the non-dimensional thickness of the denser layer ξ_2 satisfies the specific energy equation (see (A.7))

$$\xi_2^2-\xi_2^3=\varepsilon q_2^2/\underline{b}^2 \qquad \text{at } x \ge x_*, \qquad (1)$$

and

$$(1-\delta(1-\varepsilon))\xi_2^2-\xi_2^3=q_2^2/\underline{b}^2 \qquad \text{at } x \le x_*. \qquad (2)$$

This equation determines the free surface at $x \ge x_*$ and the interface at $x \le x_*$ as functions of the non-dimensional channel width, $\underline{b}(x)=b(x)/b_0$, in implicit form.

The non-dimensional free surface elevation $\xi=\zeta/\varepsilon H$ can be expressed in terms of ξ_2 as follows (see A.5)

$$\varepsilon\xi=1-\xi_2 \qquad \text{at } x \ge x_*. \qquad (3)$$

At $x < x_*$ the free surface is a horizontal plane.

The non-dimensional velocity of the denser fluid $v_2= u_2/c$ is (see (A.1))

$$v_2=\sqrt{\xi} \qquad \text{at } x > x_*, \qquad (4)$$

and (see (A.3))

$$v_2=\sqrt{\xi_1-\delta} \qquad \text{at } x \le x_*, \qquad (5)$$

where

$$\xi_1=1+\varepsilon\delta-\xi_2 \qquad \text{at } x \le x_*. \qquad (6)$$

One can see that $v_2(x)$ monotonically increases from $v_2(\infty)=0$ to $v_2(-\infty)=\sqrt{1-(1-\varepsilon)\delta}$.

Table 1

	q_1	q_2	ξ_{20}	ξ_{10}
regime 1 $1/\varepsilon \ge -\delta \ge 1/3\varepsilon$	0	$2/\sqrt{27\varepsilon}$	2/3	-
regime 2 $1/3\varepsilon \ge -\delta \ge 1/(2+\varepsilon)$	0	$(1+\varepsilon\delta)\sqrt{-\delta}$	$1+\varepsilon\delta$	0

regime 3 $1/(2+\varepsilon)\geq-\delta\geq0$	0	$2((1-\delta(1-\varepsilon))/3)^{3/2}$	$2(1-\delta(1-\varepsilon))/3$	$(1+\delta(2+\varepsilon))/3$
regime 4 $0\leq\delta\leq1$	$\delta A/\sqrt{1-\varepsilon\delta}$	$(1-\delta)A$	$\kappa_0\xi_{10}$	$2(2+\alpha)/(3(\kappa_0+2+\alpha))$
regime 5 $1\leq\delta\leq 3/(2+\varepsilon)$	$2(\delta/3)^{3/2}$	0	$1-\delta(2+\varepsilon)/3$	$2\delta/3$
regime 6 $3/(2+\varepsilon)\leq\delta\leq(1+2\varepsilon)/3\varepsilon$	$(1-\varepsilon\delta)\sqrt{\delta-1}/(1-\varepsilon)^{3/2}$	0	0	$(1-\varepsilon\delta)/(1-\varepsilon)$
regime 7 $(1+2\varepsilon)/3\varepsilon\leq\delta\leq1/\varepsilon$	$2/\sqrt{27\varepsilon}$	0	-	2/3

If a motionless lighter fluid completely covers a heavier fluid ($\delta = 0$) then according to Archimedes Principle the flow of the lower layer is the same as in absence of the upper layer but with reduced gravity. If $\delta = 0$ then $\xi_2(x)$ is described by equation (2) for all x. The shape of the interface is the same as the free surface for one layer flow ($\varepsilon\delta=-1$, in dimensional form $H_1=0$). However the discharge Q_2 in the case with $\delta = 0$ is $\sqrt{\varepsilon}$ times the discharge Q_2 of the one layer case with $\varepsilon\delta=-1$.

b. A lighter fluid running up over a stationary denser wedge. If the level of the lighter fluid reservoir is so high that $\delta>1$ ($H_1>H_2+\varepsilon H_1$) then the denser fluid is at rest, $u_2=0$, and the lighter fluid runs up over the wedge of the denser fluid.

The solution can be found from the Bernoulli and continuity equations and the requirement that the thickness of the lighter layer η_1 continuously decreases from H_1 to 0 (see Appendix B). The last three rows of Table 1 present q_1, q_2, ξ_{10} and ξ_{20} as functions of ε and δ for regime 5 (Fig. 1(f)), for regime 6 (Fig. 1(g)) and for regime 7 (Fig. 1(h)), respectively.

For given ε and δ the non-dimensional thickness of the lighter layer ξ_1 satisfies the specific energy equation (see (B.6))

$$(1-\xi_1)\,\xi_1^{\,2}=\varepsilon q/\underline{b}^{2} \qquad \text{for } x_* \geq x \qquad (7)$$

and

$$(\delta-\xi_1)\,\xi_1^{\,2}=q/\underline{b}^{2} \qquad \text{for } x \geq x_*,. \qquad (8)$$

The non-dimensional free surface $\xi=\zeta/\varepsilon H$ is determined by the equation (see (B.3)

$$\varepsilon\xi=1-\xi_1 \qquad \text{for } x_* \geq x,$$

and (see (B.5) (9)

$$\xi=\delta-\xi_1 \qquad \text{for } x \geq x_*.$$

The interface is determined by the equation (use the last equation and (B.4))

$$\xi_2=1-\varepsilon\delta-(1-\varepsilon)\xi_1 \qquad \text{for } x \geq x_*. \qquad (10)$$

The non-dimensional velocity of the lighter fluid $v_1=u_1/c$ is (see (B.1))

$$v_1=\sqrt{\xi}.$$

One can see that v_1 monotonically increases from 0 at $x \to \infty$ to $v_1(-\infty)=\sqrt{\delta}$.

3. **EXCHANGE FLOW**

The solution for regime 4 will be obtained in this section from the Bernoulli and continuity equations and the requirement that the thickness of the layers η_1 and η_2 continuously decrease from their maximum values to 0. The notation of row 4 in Table 1 will be explained.

a. ***Exact solution***. If $0 < \delta < 1$ ($0<H_1-H_2<\varepsilon H_1$) then both layers are in motion. The Bernoulli and continuity equations are

$$u_1^{\,2}=2g\zeta, \qquad (11)$$

$$u_1\eta_1 b=Q_1, \qquad (12)$$

$$u_2^{\,2}=2g(\zeta-H_1+H_2+\varepsilon\eta_1), \qquad (13)$$

$$u_2\eta_2 b=Q_2. \qquad (14)$$

We also have from the definition of the free surface displacement ζ (see Fig. 1e) that

$$\eta_1+\eta_2+\zeta=H_1. \qquad (15)$$

To obtain (11) and (13) we have used the boundary conditions

$u_1 \to 0, \zeta \to 0$ at $x\to-\infty$,

$u_2\to 0, \zeta\to H_1-H_2$ at $x\to\infty$.

We have already found the solution for $\delta = 0$ and $\delta = 1$ (see regimes 3 and 5 in Table 1). We can describe some properties of the exchange flow even before solving the problem. When δ increases from 0 to 1 then:

q_1 increases from 0 to $2/\sqrt{27}$, q_2 decreases from $2/\sqrt{27}$ to 0 (therefore q_1/q_2 increases from 0 to ∞, q_1-q_2 increases from $-2/\sqrt{27}$ to $2/\sqrt{27}$);

$\eta_1(0)$ increases from $H_1/3$ to $2H_1/3$, $\eta_2(0)$ decreases from $2H_2/3$ to $H_2/3$;

$u_1(\infty)$ increases from 0 to c, $u_2(-\infty)$ decreases from c to 0.

Using non-dimensional variables (defined in section 2, see also Appendix E) one can rewrite the system (11)-(15) in the following form

$$v_1^2=\xi, \qquad (16)$$

$$v_1\xi_1\underline{b}=q_1, \qquad (17)$$

$$v_2^2=\xi-\delta+\xi_1, \qquad (18)$$

$$v_2\xi_2\underline{b}=q_2, \qquad (19)$$

$$\xi_1+\xi_2+\varepsilon\xi=1. \qquad (20)$$

For a given channel geometry b(x), relative density difference ε and relative level difference γ ($\delta=\gamma/\varepsilon$) the system of five equations (16)-(20) contains five unknown functions $\xi(x)$, $\xi_1(x)$, $\xi_2(x)$, $v_1(x)$ and $v_2(x)$ and two unknown contraction discharge coefficients q_1 and q_2. Wood (1970) showed that q_1 and q_2 can be found using the condition that the thickness η_1 and η_2 decrease smoothly from the values H_1 and H_2 to 0. It follows from his solution that q_1 and q_2 depend only on ε and δ and are the same for any contraction.

The parameter ε appears only in equation (20). Thus the Boussinesq approximation ($\varepsilon=0$, but $\delta=\gamma/\varepsilon$, $\xi=\zeta/\varepsilon H$ and $c^2=2g\varepsilon H$ remain finite) and the rigid lid approximation ($\xi_1+\xi_2=1$) are identical for exchange flow. Note that if instead of a free surface we have a rigid lid at z=H, then one can determine the function ζ as $\zeta=((p(-\infty,H)-p(x,H))/(g\rho_1)$ and δ as $\delta=((p(-\infty, H)-p(\infty,H))/(\varepsilon gH\rho_1)$ and use the solution in the Boussinesq approximation obtained below (here p(x,H) is the pressure on the lid).

From (20) and the boundary conditions we have

$$0\le\xi_1\le\xi_1(-\infty)=1 \text{ and } 0\le\xi_2\le\xi_2(\infty)=1-\varepsilon\delta.$$

It follows from (16) and (18) that $v_1= v_2$ at the point where $\xi_1=\delta$,

$$0\le v_1\le\sqrt{\delta}=v_1(\infty) \text{ and } 0\le v_2\le\sqrt{1-\delta}=v_2(-\infty).$$

Eliminating v_1,v_2 and ξ from the system (16)-(20) we get

$$\xi_1+\xi_2+\varepsilon q_1^2/\underline{b}^2\xi_1^2=1 \qquad (21)$$

$$\delta\xi_2+q_2^2/\underline{b}^2\xi_2^2=(1-\delta)\xi_1+q_1^2(1-\varepsilon\delta)/\underline{b}^2\xi_1^2 \qquad (22)$$

Equation (22) can be called the specific energy equation for an exchange flow. The equations (21) and (22) correspond to equations (15) and (16) in Wood (1970).

Let us rewrite the equation (22) in the form

$$q_2^2-q_1^2(1-\varepsilon\delta)\kappa^2=\xi_1\xi_2^2(1-\delta-\delta\kappa)\underline{b}^2, \qquad (23)$$

where $\kappa(x)$ is the ratio of layer thicknesses

$$\kappa(x)=\xi_2(x)/\xi_1(x). \qquad (24)$$

Because $\kappa(x)$ changes continuously from 0 to ∞, there must be such a point x_v where

$$\kappa(x_v) \equiv \kappa_v = (1-\delta)/\delta. \qquad (25)$$

So, by definition, x_v is the point where $\xi_2(x)/\xi_1(x)=(1-\delta)/\delta$. Putting $x=x_v$ in (23) we get the relationship between q_1 and q_2, which corresponds to equation (26) in Wood (1970),

$$\sqrt{1-\varepsilon\delta}\, q_1/q_2=\delta/(1-\delta). \qquad (26)$$

This relationship allows us to introduce $A(\varepsilon,\delta)$ such that

$$q_1\sqrt{1-\varepsilon\delta}=A\delta,\ q_2=A(1-\delta). \qquad (27)$$

This leads to

$$\sqrt{1-\varepsilon\delta}\,q_1+q_2=A, \qquad (28)$$

$$\sqrt{1-\varepsilon\delta}\,q_1-q_2=A(2\delta-1) \qquad (29)$$

$$q_1/q_2=\delta/((1-\delta)\sqrt{1-\varepsilon\delta}) \qquad (30)$$

Substituting (27) into (16)-(20) after some algebra yields

$$\sqrt{1-\varepsilon\delta}\,v_1=\delta\xi_2/s, \qquad (31)$$

$$v_2=(1-\delta)\xi_1/s, \qquad (32)$$

$$\xi_1\xi_2\underline{b}=As, \qquad (33)$$

$$(1-\delta)\xi_1^2+\xi_1\xi_2+\xi_2^2\delta/(1-\varepsilon\delta)=(1-\delta)\xi_1+\delta\xi_2, \qquad (34)$$

where

$$s^2=(1-\delta)\xi_1+\delta\xi_2. \qquad (35)$$

From (31) and (32) we get

$$\sqrt{1-\varepsilon\delta}\,v_1+v_2=s,\ \sqrt{1-\varepsilon\delta}\,v_1-v_2=(\delta-\xi_1-\varepsilon\delta\xi)/s,\ \sqrt{1-\varepsilon\delta}\,v_1=\kappa v_2/\kappa_V \qquad (36)$$

In particular $\sqrt{1-\varepsilon\delta}\,v_1(x_V)=v_2(x_V)$ (see equation (22) in Wood, 1970). If $\delta=1/2$ one can see from (25) that $\eta_1(x_v)=\eta_2(x_v)$ and from (26) that $H_2Q_1^2=H_1Q_2^2$. From (26) we get that $Q_1=Q_2$ when $\varepsilon=(1-2\delta)/(\delta(1-\delta)^2)$ or approximately $\delta\approx0.5-0.07\varepsilon$ for $0\le\varepsilon\le1$.

If we add to the system (31)-(32), (34)-(35) one more condition, for example, $v_1=v_2$ or $\xi_1=\xi_2$ (each of which occurs at some position), we get five equations. These equations allow us to find all five functions $v_1(\varepsilon,\delta)$, $v_2(\varepsilon,\delta)$, $\xi_1(\varepsilon,\delta)$, $\xi_2(\varepsilon,\delta)$ and $s(\varepsilon,\delta)$ (see Appendix C) which are independent of $\underline{b}(x)$ and do not require knowledge of $A(\varepsilon,\delta)$.

The analytical solution of the system (31)-(35) in parametric form, which expresses ξ_1, ξ_2, v_1, v_2 and s in terms of b(x), is presented in Appendix D. The analytical expression for $A(\varepsilon,\delta)$ will be obtained below from the condition that the thicknesses η_1 and η_2 are smooth functions. At this time we would like to point out that the expressions (31) and (32) for the velocities v_1 and v_2 in terms of ξ_1 and ξ_2 and the quadratic equation (34), from which one can express ξ_1 in terms of ξ_2, contain neither $A(\varepsilon,\delta)$ nor the channel width b(x). We see that the functions $v_1(\xi_2)$, $v_2(\xi_2)$ and $\xi_1(\xi_2)$ do not depend on channel geometry. Solving the quadratic equation (34) and using (35) one gets from (33) the algebraic expression for $b(\xi_2)$, which gives the interface as function of the channel width in implicit form (see Fig. 1(e)).

In the system (31)-(35) the parameter ε occurs only in the combination $1-\varepsilon\delta$. Therefore the Boussinesq approximation can be used for entire range of ε ($0<\varepsilon<1$) if $\gamma<<1$.

To find $A(\varepsilon,\delta)$ we use (34) to rewrite equation (33) as

$$A^2\delta(\kappa+\kappa_v)(1+\kappa+\varepsilon\delta\kappa^2/((1-\varepsilon\delta)(\kappa+\kappa_v)))^3/\kappa^2=\underline{b}^2.$$

The right side of this equation has a minimum at x=0. Therefore the left side must also have a minimum at x=0. This leads to

$$2\kappa_0^2-\kappa_0(1-\kappa_v)-2\kappa_v+2\varepsilon\kappa_0^2(\kappa_0+2\kappa_v)/((1-\varepsilon+\kappa_v)(\kappa_0+\kappa_v))=0. \qquad (37)$$

Here $\kappa_0=\kappa(0)$ is the ratio of layer thicknesses at the narrowest section which decreases from 2 to $(1-\varepsilon)/2$ when δ increases from 0 to 1.

Introducing the parameter $\alpha=\kappa_0/\kappa_v$ one can find from (33), (34) and (37) the following expression for $A(\varepsilon,\delta)$ in parametric form

$$A^2=8(\kappa_0+\alpha)\kappa_0/(27(1+\alpha)(1+\kappa_0/(2+\alpha))^3), \qquad (38)$$

$$\delta=\alpha/(\kappa_0+\alpha), \qquad (39)$$

$$\kappa_0=(1-\alpha^2+.5\varepsilon\alpha(1-\alpha)/(1+\alpha)+ \qquad (40)$$

$$+\sqrt{(1+\alpha(1-\varepsilon/2)+\alpha^2)^2-6\varepsilon\alpha^2-\varepsilon^2\alpha^3/(1+\alpha)^2})/(1+2\alpha),$$

which completes the solution.

One can see from (39) that α varies over the interval $(0,\infty)$ when δ changes from 0 to 1. The fourth row of Table 1 presents the contraction discharge coefficients q_1 and q_2 and non-dimensional thicknesses ξ_{10} and ξ_{20}

as functions of ε and δ in parametric form (α is the parameter and κ_0 is given by (40)) for regime 4 (Fig. 1(e)). The velocities $v_1(0)$ and $v_2(0)$ can be found from relations

$$v_1(0)=q_1/\xi_{10} \quad \text{and} \quad v_2(0)=q_2/\xi_{20}.$$

Figures 2-5 illustrate the solutions presented in the Table 1. Fig. 2 shows the graphs q_1^2 and q_2^2 as functions of δ for a wide range of the parameter ε (cf. Wood, 1970, Fig. 6). If the lighter fluid is air and the denser fluid is water then $\varepsilon\approx0.9987$. For $\varepsilon=0.9987$ regimes 5, 6 and 7 (water is motionless) exist when $H_2<0.0013H_1$. For $\varepsilon\rightarrow1$ regimes 1, 2 and 3 merge into regime 1 at $-1<\delta<0$ and regimes 5, 6 and 7 merge to one point at $\delta=1$. Remember that δ changes in the interval $[-1/\varepsilon,1/\varepsilon]$. The graphs $q_1^2(\delta)$ and $q_2^2(\delta)$ are presented in Fig. 2 in their entirety for all seven regimes for $\varepsilon=1$ ($-1<\delta<1$), $\varepsilon=0.75$ ($-4/3<\delta<4/3$) and $\varepsilon=0.5$ ($-2<\delta<2$).

Fig. 3 shows the graphs $q_1(\delta)$ and $q_2(\delta)$ (cf. Wood, 1970, Fig. 6) for exchange flow only and the values of ε as in Wood (1970). If $\varepsilon=0$ the graphs $q_1(\delta)$ and $q_2(\delta)$ are symmetrical on either side of the line $\delta=0.5$. When ε increases then $q_1(\delta)$ slightly increases and $q_2(\delta)$ slightly decreases; both the value of δ for which $q_1=q_2=q_=$ and the value of $q_=$ slightly decrease.

Fig. 4 shows graphs ξ_{10}, ξ_{20} and $\zeta(0)/H$ as functions of δ for regimes 3, 4 and 5 and the values of ε as in Wood (1970). If $\varepsilon=0$ the graphs $\xi_{10}(\delta)$ and ξ_{20} are symmetrical on either side of the line $\delta=0.5$ and $\zeta(0)/H=0$. Note that the graphs $\eta_{20}(\varepsilon,\delta)/H_2$ will be closer to each other for different ε than the graphs $\eta_{20}(\varepsilon,\delta)/H_1$. The graphs ξ_{10} and ξ_{20} as functions of δ and Wood's (1970) experimental data are shown on Fig. 5.a and 5.b, correspondingly. The agreement between the experimental points and theoretical curves is very good. Figures 4 and 5 correspond to Figures 9 and 10 in Wood (1970) which present ξ_{10} and ξ_{20} as functions of $H_2/H_1=1-\varepsilon\delta$.

b. ***Boussinesq approximation***. In most natural flows both ε and γ are very small. If $\varepsilon>\gamma>0$, an exchange flow occurs and the Boussinesq approximation can be used. In other words one can put $\gamma=0$ ($H_1=H_2$) and $\varepsilon=0$ ($\rho_1=\rho_2$) everywhere except the expressions $c^2=2\varepsilon gH$, $\xi=\zeta/(\varepsilon H)$ and $\delta=\gamma/\varepsilon$. For exchange flow $0<\delta<1$, therefore the condition $\gamma<<1$ follows from the condition $\varepsilon<<1$.

In the Boussinesq approximation it is convenient to use the ratio of layer thicknesses at the narrowest section κ_0 ($0.5<\kappa_0<2$) as the parameter. The complete solution of the problem is (putting $\varepsilon=0$ in (40) and using (38)-(39) and the fourth row of Table 1)

$$\delta=(1-\kappa_0/2)/(1-\kappa_0+\kappa_0^2), \qquad (41)$$

$$A^2=2\kappa_0(1+\kappa_0^3)/(1+\kappa_0)^5, \qquad (42)$$

$$q_1^2=2\kappa_0(1-\kappa_0/2)^2/(1+\kappa_0^3)(1+\kappa_0)^3, \qquad (43)$$

$$q_2^2=2\kappa_0^3(\kappa_0-1/2)^2/(1+\kappa_0^3)(1+\kappa_0)^3, \qquad (44)$$

$$\xi_{10}=1/(1+\kappa_0),\ \ \xi_{20}=\kappa_0/(1+\kappa_0). \qquad (45)$$

Note that one can get an explicit expression for $\kappa_0(\delta)$ from quadratic equation (41) and obtain $A(\delta)$ in the explicit form

$$6A^2(\delta) = (1+2\sqrt{1-3\mu^2})/(1+\sqrt{1-3\mu^2})^2, \qquad (46)$$

where $\mu=\delta-1/2$. Because $\kappa_0=1/\xi_{10}-1$ one can rewrite the solution (41)-(45) using ξ_{10} as the parameter.

Using $\underline{x}$ as a parameter ($-\infty<\underline{x}<\infty$) we have (putting $\varepsilon=0$ in Appendix D)

$$2\xi_1=1-\tanh\underline{x}, \qquad (47)$$

$$2\xi_2=1+\tanh\underline{x}, \qquad (48)$$

$$\sqrt{2}\,v_1=\delta\,(1+\tanh\underline{x})/(1+(2\delta-1)\tanh\underline{x})^{1/2}, \qquad (49)$$

$$\sqrt{2}\,v_2=(1-\delta)\,(1-\tanh\underline{x})/(1+(2\delta-1)\tanh\underline{x})^{1/2}, \qquad (50)$$

$$\sqrt{2}\,s=(1+(2\delta-1)\tanh\underline{x})^{1/2}, \qquad (51)$$

$$\underline{b}(x)=\sqrt{8}\,A\cosh^2\underline{x}\,(1+(2\delta-1)\tanh\underline{x})^{1/2}. \qquad (52)$$

We choose the monotonically increasing solution $\underline{x}(x)$ of equation (52). When x increases from $-\infty$ to ∞ then $\underline{x}$ increases from $-\infty$ to ∞.

In the Boussinesq approximation $x_u=x_v$. At this point (see Table in Appendix C)

$$\xi_1=\delta,\ \xi_2=1-\delta,\ \underline{x}=\tanh^{-1}(1-2\delta),\ v_1=v_2=\sqrt{\delta(1-\delta)/2},\ \underline{b}=A\sqrt{2/\delta(1-\delta)}. \qquad (53)$$

The formulas (41)-(45) and (53) give the analytic solution of the system (3.11.9)-(3.11.14) in Baines (1995).

At the point x_η where the layer thicknesses are equal ($\xi_1=\xi_2=1/2$) we have (see Table in Appendix C)

$$\underline{x}=0,\ v_1=\delta/\sqrt{2},\ v_2=(1-\delta)/\sqrt{2},\ s=1/\sqrt{2},\ \underline{b}=A\sqrt{8}. \qquad (54)$$

The formulas (41a and 41b) in Lawrence (1990) give identical expressions for $\underline{b}$ at these points.

If $\delta=1/2$ ($\kappa_0=1$) the formulas (42)-(51) become very simple,

$$A^2=1/8,\ q_1=q_2=\sqrt{2}/8,\ \xi_{10}=\xi_{20}=1/2. \qquad (55)$$

[in dimensional form $Q_1=Q_2=b_0H(\varepsilon gH)^{1/2}/4$, (Schijf and Schönfeld 1953, p.325)] and

$$2\xi_1=1-(1-1/\underline{b})^{1/2},\ 2\xi_2=1+(1-1/\underline{b})^{1/2}, \qquad \text{for } x \geq 0,$$
$$2\xi_2=1-(1-1/\underline{b})^{1/2},\ 2\xi_1=1+(1-1/\underline{b})^{1/2}, \qquad \text{for } x \leq 0, \qquad (56)$$
$$v_1=\xi_2/\sqrt{2},\ v_2=\xi_1/\sqrt{2},\ s=1/\sqrt{2}.$$

In the Boussinesq approximation $v_1+v_2=s$ (one can call s the shear), $q_1+q_2=A(\delta)$ (one can call A the exchange flow rate), $U=q_1-q_2=(2\delta-1)A(\delta)$ and $q_r=q_1/q_2=\delta/(1-\delta)$ (put $\varepsilon=0$ into (36), (28), (29) and (30) respectively). Presented above are analytical formulas for all functions shown in Figures 5-8 by Armi and Farmer (1986) (they used U as the independent variable) and in Figures 6-8 by Lawrence (1990) (he used q_r as the independent variable). $A(\delta)$ changes insignificantly ($\max(A^2)=A^2(0)=A^2(1)=4/27$, $\min(A^2)=A^2(1/2)=4/32$). Therefore the function $U(\delta)$ is close to linear and the graphs of all values as functions of U (Armi and Farmer (1986)) and as functions of δ (this paper) look similar. It is trivial that $q_2\equiv-U$ and $q_r=0$ for regimes 1-3 (because $q_1=0$), $q_1\equiv U$ and $q_r=\infty$ for regimes 5-7 (because $q_2=0$).

Table 2 is a simplified transposed version of Table 1. To get formulas for regime 4 in Table 2 we use solutions for regime 3 for $\delta=0$ and for regime 5 for $\delta=1$ and suppose that $\xi_{10}(\delta)$ and $\xi_{20}(\delta)$ are linear continuous functions for regime 4. The sign ≈ in Table 2 means that in addition to the Boussinesq approximation we also approximate $\xi_{10}(\delta)$ and $\xi_{20}(\delta)$ as linear functions and then we find $v_{10}(\delta)$ and $v_{20}(\delta)$ from (17) and (19) putting $A=2/\sqrt{27}$ for all δ from the interval [0,1].

Table 2

	regime 2	regime 3	regime 4	regime 5	regime 6
	$-\delta \geq 1/2$	$1/2 \geq -\delta \geq 0$	$0 \leq \delta \leq 1$	$1 \leq \delta \leq 1.5$	$1.5 \leq \delta$
q_1	0	0	δA	$2(\delta/3)^{3/2}$	$(\delta-1)^{1/2}$
q_2	$\sqrt{-\delta}$	$2((1-\delta)/3)^{3/2}$	$(1-\delta)A$	0	0
ξ_{10}	0	$(1+2\delta)/3$	$\approx(1+\delta)/3$	$2\delta/3$	1
ξ_{20}	1	$2(1-\delta)/3$	$\approx(2-\delta)/3$	$1-2\delta/3$	0
v_{10}	0	0	$\approx 2\delta/(1+\delta)\sqrt{3}$	$(\delta/3)^{1/2}$	$(\delta-1)^{1/2}$
v_{20}	$\sqrt{-\delta}$	$((1-\delta)/3)^{1/2}$	$\approx 2(1-\delta)/(2-\delta)\sqrt{3}$	0	0

If $\varepsilon \ll 1$ then, until $\varepsilon\delta \ll 1$, the Boussinesq approximation can be also used for regimes 2, 3, 5 and 6.

4. **Some examples of application of the theory.**

The essential quantitative problem is the determination of the net volume exchange rate $V'=Q_1-Q_2$ and the net mass exchange rate $M'=\rho_2Q_2-\rho_1Q_1$ if we know ε and δ and vice versa (see for example Bryden and Kinder (1991), Hogg and Huang (1995), Chapter 4). Between M' and V' there is a simple relation

$$M'/\rho_2=\varepsilon Q_1-V'=Q_2-(1-\varepsilon)Q_1. \qquad (57)$$

If $M'=0$ we have $\varepsilon=1-Q_2/Q_1=1-q_2/q_1$ and $V'=\varepsilon Q_1$. If $V'=0$ we have $Q_1=Q_2$ and $\varepsilon=M'/(\rho_2Q_1)$. The relations (57) are correct for any exchange flow. For the case of pure contraction, considered in this paper, we have using (27)

$$V'=\sqrt{2\varepsilon}\,A(\delta)Q_0(\,2\delta-1+\varepsilon\delta^2/(1+\sqrt{1-\varepsilon\delta}-\varepsilon\delta)), \qquad (58)$$
$$M'/\rho_2=\sqrt{2\varepsilon}\,A(\delta)Q_0(\,1-2\delta+\varepsilon\delta(1+\sqrt{1-\varepsilon\delta}-\delta)/(1+\sqrt{1-\varepsilon\delta}-\varepsilon\delta)). \qquad (59)$$

Here $Q_0{}^2=gb_0{}^2H^3$. For any given M' and V' one can find ε and δ from (58) and (59). If both M'/Q_0 and V'/Q_0 are order $\varepsilon^{1/2}$ then $\delta-0.5$ is not small for small ε. If M' or V' is zero then $2\delta=1+O(\varepsilon)$. For $M'=0$ we have

$$2\delta=1+\varepsilon\delta^2(2-\delta-\varepsilon)/(1-\varepsilon\delta)=1+3\varepsilon/8+O(\varepsilon^2)$$
$$\varepsilon^{3/2}=(1-\varepsilon)V'/(\sqrt{2}(1-\delta)A(\delta)Q_0)=4V'/Q_0+O(\varepsilon^{5/2}).$$

In the quasi-steady approximation we assume that $\varepsilon(t)$, $\delta(t)$, $M'(t)$ and $V'(t)$ slowly vary in time so that the equations (58) and (59) are correct.

Suppose that $\delta(t)=\delta_0+\delta_1\sin(2\pi t/T)$ (due to tides), $V'(t)=E$ (due to evaporation from the basin with the heavier fluid) and $\langle M'(t)\rangle=0$, here <...> means average over period T. Let δ_1 and E be known constants, then ε and δ_0 are unknown constants. One can find ε and δ_0 from

$$\langle(1-\delta)A(\delta)\rangle/(1-\varepsilon)=\langle\delta A(\delta)/(1-\varepsilon\delta)^{1/2}\rangle=E/(\varepsilon^{3/2}\sqrt{2}\,Q_0). \qquad (60)$$

For quasi-steady applications of this theory it is important that the so called barotropic transport $U=q_1-q_2$ is not zero ($U= -2/\sqrt{27}$) when the levels in the reservoirs are equal ($\delta=0$) but $U=0$ when $\delta=1/2$. For instance if the level of the heavier fluid reservoir changes periodically $H_2=H_1(1-\varepsilon(a+f(t))$ (here H_1 and a are constants, f(t) is a periodic function with period T and $\langle f(t)\rangle=0$), then $\delta=a+f(t)$ and $U=2A(\delta)(a-0.5+f(t))$ are periodic functions with the same period T. Note that $\langle U(t)\rangle=0$ (this means $\langle q_1\rangle=\langle q_2\rangle$) if a=1/2 and f(t) is an even function. But generally $\langle U(t)\rangle\neq 0$ and depends on a and the amplitude and behavior of f(t). Using (46) one can easily calculate $\langle U(t)\rangle$ for any particular a and f(t).

In Helfrich's (1995) experiment a small tank filled with a fluid of density ρ_1 and volume V_0 begins to vertically oscillate with a displacement $a_0\sin(2\pi t/T)$ in a large basin filled with a fluid of density ρ_2. Because the free surface area of the small tank, S, was small in this experiment ($Sa_0\sim Q_0T\sqrt{\varepsilon}$), the change of the level in the small basin due to flux through the contraction must be taken into consideration. Neglecting changes of density and volume in the large basin we have

$$d\delta/dt \approx (2\pi(a_0/T)\cos(2\pi t/T)+(Q_2- Q_1)/S)/(H\varepsilon) \qquad (61)$$

$$d\varepsilon/dt \approx -\varepsilon\, Q_2/V \qquad (62)$$

The first term in the right side of the equation (61) describes the change in H_2 due to oscillations of the small tank, the second term describes the change in H_1 due to the change of the fluid volume in the small tank. Equation (62) describes the change of ε due to the net mass exchange M. We also can put $\varepsilon=\varepsilon(0)$ and $V=V(0)$ in the right sides of the equations (61) and (62). We get two equations for $\varepsilon(t)$ and $\delta(t)$ which can be easily solved with initial conditions $\varepsilon(0)=\varepsilon_0$, $\delta(0)=0$.

The above examples illustrate some applications of the theory to steady and quasi-steady flows.

The equations (58) - (60) are valid for regime 4. If $\delta(t)$ becomes negative and/or larger than 1, then these equations must be extended using formulas for q_1 and q_2 for regimes 2, 3, 5 and 6.

5. **Discussion**

a. Comparison with one layer flow. When the fluid on both sides of the contraction is the same density then only one-directional flow is possible. A steady flow occurs if $2H_1>3H_2$ or $2H_2>3H_1$. When the fluids on both sides of the contraction are of different densities exchange flow takes place only if the level of the lighter fluid is slightly higher than the level of the denser fluid (in the case of the rigid lid, the pressure on the lid far from the contraction on the side of the lighter fluid is slightly higher than the pressure on the rigid lid far from contraction on the side of the denser fluid). Precisely, the relative level difference γ must be less than the relative density difference ε (relative pressure difference must be less then the relative density difference in the case of the rigid lid). The maximum velocity of exchange flow ($=\sqrt{2\varepsilon gH}$) is smaller than the maximum velocity of steady one layer flow ($=\sqrt{2gH}$).

When the fluid on both sides of the contraction is the same density the flux through the contraction can be zero if the levels in the basins are equal ($H_1=H_2$). In the case of different densities the exchange flow rate Q_1+Q_2 is always larger than some positive value. In the Boussinesq (or rigid lid) approximation $Q_1+Q_2\geq Q_{min}=\sqrt{\varepsilon}\,Q_0/2$ and reaches this minimum value when $\delta=1/2$. Here $Q_0^2=gb_0^2H^3$. The discharge for one layer steady flow through a contraction is $2Q_0/\sqrt{27}$.

b. Many features of real exchange flows were neglected in this paper. We have found the discharge coefficients q_1 and q_2 for steady flow using a simple model which does not include effects such as friction, decreasing channel width with depth, changing depth along a channel, etc.

The expressions $Q_{1,2}^2= q_{1,2}g'b_0^2H^3$ can always be written by replacing $q_{1,2}$ with $q_{1,2ef}$. To take into account the effects of friction and/or decreasing channel width with depth on the discharge coefficients q_1 and q_2 we introduce correction coefficients C_1 and C_2 such that

$$q_{1ef}=C_1q_1, \quad q_{2ef}=C_2q_2, \qquad (63)$$

where q_1 and q_2 are the discharge coefficients from Table 1. Corresponding modifications must be made in (58) - (60).

If the channel width b(x,z) increases from the bottom to the free surface and b_0 in $Q_{1,2}{}^2= q_{1,2}g'b_0{}^2H^3$ is the minimum width of the free surface, then $1>C_1(\delta)>C_2(\delta)$. For some b(x,z) the correction coefficients $C_1(\delta)$ and $C_2(\delta)$ can be found analytically.

To include effects of friction, some empirical friction coefficients are required in the momentum equations. Alternatively, it is reasonable to introduce empirical correction coefficients $C_1<1$ and $C_2<1$ in (63) to take account of the resistance losses (we assume that, similar to one layer flow, see Bakhmeteff, 1932, pp. 42-43, the friction reduces discharges of both layers). These coefficients are different for different contractions.

Due to friction the depth of the interface and the depth of the zero velocity line can differ significantly in real flow, especially for regimes 2,3, 5 and 6 (see for example Arita and Jirka, 1987, Fig.2). This and other effects of friction are outside the scope of our consideration.

When we model continuously stratified flow by two-layer flow the correct choice of ρ_1 and ρ_2 is very important. A small difference in ρ_1 and/or ρ_2 can lead to significant difference in ε and as a result a significant differences in δ, q_1 and q_2.

The key relation of the theory (26) was obtained under the assumption that b(x) infinitely increases far from the contraction. Let a maximal width of the lighter fluid reservoir be a constant b_- at $x<x_-<0$ and a maximal width of the heavier fluid reservoir be a constant b_+ at $x>x_+>0$. One can use the solution obtained in section 3 in the interval $[x_-,x_+]$ and take $\eta_i(x)=\eta_i(x_-)$, $u_i(x)=u_i(x_-)$ at $x<x_-$ and $\eta_i(x)=\eta_i(x_+)$, $u_i(x)=u_i(x_+)$ at $x>x_+$ (i=1,2). Note that $\eta_i(x)$ and $u_i(x)$ can be arbitrary constants in the channel with constant width.

c. Concluding remarks. Exchange flow through a contraction takes place because the fluids on either side of the contraction are of different densities and levels. The important result is that for small ε the existence of an exchange flow and the solution depend only on the ratio of the relative reservoir level difference γ to the relative density difference ε, $\delta=\gamma/\varepsilon$. In contrast to previous authors who presented results of numerical calculations we obtained the complete analytical solution of the problem. In section 4 we demonstrated application of the analytical theory for the solution of some practical problems. The choice of δ as an independent parameter is more successful than the choice of "net discharge" $U=q_1-q_2$ or the discharge ratio $q_r=q_1/q_2$ to obtain an analytical solution for all regimes and to understand the underlying physics.

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Appendix A. Denser fluid plunging under a stationary lighter fluid

If $\delta\leq 0$ then there is no flow of lighter fluid (Fig. 1 b-d). The Bernoulli and continuity equations for the denser fluid are

$$u_2{}^2=2g\zeta, \qquad (A.1)$$

$$u_2\eta_2 b=Q_2, \qquad (A.2)$$

in the region upstream of the point ($x>x_*$) where the flowing layer plunges under the stationary fluid; and

$$u_2{}^2=2g(H_2-H_1+\varepsilon\eta_1), \qquad (A.3)$$

$$u_2\eta_2 b=Q_2, \qquad (A.4)$$

in the region downstream of the plunging point ($x<x_*$). We also have

$$\eta_2+\zeta=H_2 \qquad \text{at } x>x_*, \qquad (A.5)$$
$$\eta_1+\eta_2=H_1 \qquad \text{at } x<x_*. \qquad (A.6)$$

Here u_2, η_2 and Q_2 are the velocity, thickness and discharge of the denser fluid, ζ is the free surface displacement, η_1 is the thickness of the lighter layer.

Eliminating u_2 and ζ from (A.1), (A.2) and (A.5) we get

$$(H_2-\eta_2)\eta_2^2=Q_2^2/2gb^2 \qquad \text{at } x>x_*. \qquad (A.7a)$$

Eliminating u_2 and η_1 from (A.3), (A.4) and (A.6) we get

$$(H_1-\eta_2+(H_2-H_1)/\varepsilon))\eta_2^2=Q_2^2/2\varepsilon gb^2 \qquad \text{at } x<x_*. \qquad (A.7b)$$

We are looking for a continuous solution of the equation (A.7) with boundary conditions

$$\eta_2\to 0 \quad \text{at } x\to-\infty, \quad \eta_2\to H_2 \quad \text{at } x\to\infty.$$

When the plunging point is downstream of the position of minimum width ($x_*<0$, regime 1), the requirement that the left side of the equation (A.7a) must have a maximum at x=0, because the right side has a maximum at x=0, gives

$$\eta_{20}=2H_2/3, \quad Q_2^2=g\eta_{20}^3b_0^2. \qquad (A.8)$$

The last expression gives $u_{20}^2=g\eta_{20}$. Putting $\eta_2(x_*)=H_1$ and Q_2 from (A.8) into (A.7a) we get the equation for the position of the plunging point x_*

$$b^2(x_*)/b_0^2=4H_2^3/(27H_1^2(H_2-H_1)). \qquad (A.9)$$

Therefore regime 1 exists when $2H_2/3>H_1>0$.

When the plunging point is upstream of the position of minimum width ($x_*>0$, regime 3), the requirement that the left side of the equation (A.7b) must have a maximum at x=0, because the right side has a maximum at x=0, gives

$$\eta_{20}=2(H_1+(H_2-H_1)/\varepsilon)/3, \quad Q_2^2=\varepsilon g\eta_{20}^3b_0^2. \qquad (A.10)$$

The last expression gives $u_{20}^2=\varepsilon g\eta_{20}$. Putting $\eta_2(x_*)=H_1$ and Q_2 from (A.10) into (A.7b) we get the equation for the position of the plunging point x_*

$$b^2(x_*)/b_0^2=4(H_1+(H_2-H_1)/\varepsilon)^3/(27H_1^2(H_2-H_1)). \qquad (A.11)$$

Therefore regime 3 exists when $H_2>H_1>H_2/(1+\varepsilon/2)$.

When the plunging point is at the position of minimum width ($x_*=0$, regime 2) then $\eta_{20}=H_1$ (see Fig.1c). Putting $\eta_{20}=H_1$ in (A.7a) we have

$$Q_2^2=2g(H_2-H_1)H_1^2b_0^2.$$

This gives $u_{20}^2=2g(H_2-H_1)$.

The layer thickness η_2 decreases and velocity u_2 increases in the flow direction. If one defines the critical velocity as $(g\eta_{20})^{1/2}$ for regime 1, as $(2g(H_2-\eta_{20}))^{1/2}$ for regime 2 and as $(\varepsilon g\eta_{20})^{1/2}$ for regime 3 then "the flow upstream of the minimum width is subcritical and downstream of it is supercritical" (Wood (1970), § 2.2(a)).

Appendix B. **Lighter fluid running up on a stationary denser fluid**

If $1\le\delta$ then there is no flow of denser fluid (Fig. 1 f-h). The Bernoulli and continuity equations for the lighter fluid are

$$u_1^2=2g\zeta, \qquad (B.1)$$
$$u_1\eta_1 b=Q_1, \qquad (B.2)$$

We also have

$$\eta_1+\zeta=H_1 \qquad \text{at } x<x_*, \qquad (B.3)$$

in the region upstream of the plunging point where the flowing layer runs on the stationary fluid;

$$\zeta+\eta_1+\eta_2=H_1 \qquad \text{at } x>x_*, \qquad (B.4)$$
$$\zeta+\varepsilon\eta_1=H_1-H_2 \qquad \text{at } x>x_*. \qquad (B.5)$$

in the region downstream of the plunging point. The last equation is the condition of no motion of the denser fluid. From (B.4) and (B.5) we have $(1-\varepsilon)\eta_1(x_*)=H_2$. Here u_1, η_1 and Q_1 are the velocity, thickness and discharge of the lighter fluid.

Eliminating u_1 and ζ from (B.1), (B.2) and (B.3) we get

$$(H_1-\eta_1)\eta_1^2=Q_1^2/2gb^2 \quad \text{at } x<x_*. \qquad (B.6a)$$

Eliminating u_1 and ζ from (B.1), (B.2) and (B.5) we get

$$(-\eta_1+(H_1-H_2)/\varepsilon))\eta_1^2=Q_1^2/2\varepsilon gb^2 \quad \text{at } x>x_*. \qquad (B.6b)$$

We are looking for a continuous solution of the equation (B.6) with boundary conditions

$$\eta_1 \to 0 \quad \text{at } x\to\infty, \quad \eta_1 \to H_1 \quad \text{at } x\to -\infty.$$

When the plunging point is upstream of the position of minimum width ($x_*>0$, regime 7), the requirement that the left side of the equation (B.6a) must have a maximum at x=0, because the right side has a maximum at x=0, gives

$$\eta_{10}=2H_1/3, \quad Q_1^2=g\eta_{10}^3b_0^2. \qquad (B.7)$$

The last expression gives $u_{10}^2=g\eta_{10}$. Putting $\eta_1(x_*)=H_2/(1-\varepsilon)$ and Q_1 from (B.7) into (A.6a) we get the equation for the position of the plunging point

$$b^2(x_*)/b_0^2=4(1-\varepsilon)^2H_1^3/(27H_2^2(H_1-H_2/(1-\varepsilon))). \qquad (B.8)$$

Therefore regime 7 exists when $2(1-\varepsilon)H_1/3>H_2>0$.

When the plunging point is upstream of the position of minimum width ($x_*>0$, regime 5), the requirement that the left side of the equation (B.6b) must have a maximum at x=0, because the right side has a maximum at x=0, gives

$$\eta_{20}=2(H_1-H_2)/3\varepsilon, \quad Q_2^2=\varepsilon g\eta_{20}^3b_0^2. \qquad (B.9)$$

The last expression gives $u_{20}^2=\varepsilon g\eta_{20}$.

Putting $\eta_1(x_*)=H_2/(1-\varepsilon)$ and Q_1 from (B.9) into (B.6b) we get the equation for the position of the plunging point

$$b^2(x_*)/b_0^2=4(1-\varepsilon)^2((H_1-H_2)/\varepsilon)^3/(27H_2^2(H_1-H_2/(1-\varepsilon))). \qquad (B.10)$$

Therefore regime 5 exists when $(1-\varepsilon)H_1>H_2>(1-\varepsilon)H_1/(1+\varepsilon/2)$.

When the plunging point is at the position of minimum width ($x_*=0$, regime 6) we put $\eta_{10}=H_2/(1-\varepsilon)$ into (B.6a) to find

$$Q_1^2=2g(H_1-H_2/(1-\varepsilon))(H_2/(1-\varepsilon))^2b_0^2.$$

This gives $u_{10}^2=2g(H_1-H_2/(1-\varepsilon))$.

Appendix C. The solution at $x=x_v$, $x=x_u$ and $x=x_\eta$.

Let the point where the layer velocities are equal ($u_1=u_2$) be defined as x_u and the point where the layer thicknesses are equal ($\eta_1=\eta_2$) be defined as x_η. The solution of the system (31)-(32), (34)-(35) has a simple form at points $-\infty$, x_v, x_u, x_η and ∞ (see Table). To calculate the channel width at these points from (33) we need to know $A(\varepsilon,\delta)$.

"To measure the *degree of rapidity* of flow" Bakhmeteff (1932), pp. 64-65, introduced "the *kinetic flow factor"* $F_i(x)=2v_i^2/\xi_i(x)$ (i=1,2) which is "twice the ratio of the kinetic energy head to the potential energy head". Recent authors use the notation $F=Fr^2$, where Fr is called the local Froude number (Baines (1995), p.38 and §1.4). Wood (1970, Fig.4) showed that the combination $F_1+F_2-\varepsilon F_1F_2$ is equal to unity at the position of minimum width and some other point and less than unity between these points. He called them "points of control".

From (31) and (32) we get

$$(1-\varepsilon\delta)F_1=2\delta\kappa^2/(\kappa+\kappa_v), \quad F_2=2(1-\delta)/(\kappa+\kappa^2/\kappa_v).$$

In particular $F_1+F_2-\varepsilon F_1F_2=1$ at $x=x_v$. Equation (37) ensures that $F_1+F_2-\varepsilon F_1F_2=1$ at x=0.

Table

x	$-\infty$	x_v	x_u	x_η	∞
ξ	0	$(1-\delta)\delta/(2-\varepsilon\delta-\varepsilon\delta^2)$	$\delta(1-\delta)/(1+\sqrt{1-\varepsilon\delta})$	$\delta^2/(2-2\varepsilon\delta+\varepsilon\delta^2)$	δ
ξ_1	1	$\delta(1-\varepsilon\xi)$	δ	$(1-\varepsilon\xi)/2$	0
ξ_2	0	$(1-\delta)(1-\varepsilon\xi)$	$(1-\delta)\sqrt{1-\varepsilon\delta}$	$(1-\varepsilon\xi)/2$	$1-\varepsilon\delta$

v_1^2	0	ξ	ξ	ξ	δ
v_2^2	$1-\delta$	$\xi(1-\varepsilon\xi)$	ξ	$(1-\delta)^2(1-\varepsilon\xi)/2$	0
s^2	$1-\delta$	$2(1-\delta)\delta(1-\varepsilon\delta)$	$\delta(1-\delta)(1+\sqrt{1-\varepsilon\delta})$	$(1-\varepsilon\xi)/2$	$\delta(1-\varepsilon\delta)$
$\underline{b}^2(x)$	∞	$2A^2/(\delta(1-\delta)(1-\varepsilon\xi)^3)$	$A^2/(\xi(1-\varepsilon\delta))$	$8A^2/(1-\varepsilon\xi)^3$	∞
$(1-\varepsilon\delta)F_1$	0	$1-\delta$	$2(1-\delta)(1-\varepsilon\delta)/(1+\sqrt{1-\varepsilon\delta})$	$2\delta^2$	∞
F_2	∞	δ	$2\delta/(1+\sqrt{1-\varepsilon\delta}-\varepsilon\delta)$	$2(1-\delta)^2$	0

We see from the Table that $x_v<x_\eta$ when $\delta>0.5$ and $x_v>x_\eta$ when $\delta<0.5$ (because ξ monotonically increases with x) and that $2s^2(x)<1$ for $0<\varepsilon<1$ at points x_V, x_u and x_η. We also see that $F_1(x_V)=F_2(x_V)$ when $1-\varepsilon\delta=\kappa_V$ ($\eta_2(x_V)/H_2=\eta_1(x_V)/H_1$ and $H_1^2\rho_1=H_2^2\rho_2$ in dimensional form). This gives $\delta=1/(1+\sqrt{1-\varepsilon})$.

Wood (1970) called x_V the point of virtual control and found $\xi_1(x_V)$, $\xi_2(x_V)$, $v_1^2(x_V)$ and $v_2^2(x_V)$ (equations (22)-(25) in Wood, 1970). Lawrence (1990) noted that the solution is simple when $F_1(x_V)=F_2(x_V)$. Indeed, for $\delta=1/(1+\sqrt{1-\varepsilon})$:

(25) gives $\kappa_V=\sqrt{1-\varepsilon}$;

using (20) we get $\xi_1(x_V)=2/(3+\kappa_v)$ and $\xi_2(x_V)=2\kappa_v/(3+\kappa_v)$ (see equations (24) and (25) in Wood, 1970), (26) takes the form $q_r^2=\kappa_v^{-3}$ (the last three equations are identical to equations (44a,b) and (43) in Lawrence (1990), the graphs of $\xi_1(x_V)$, $\xi_2(x_V)$ and q_r as functions of ε are presented in Fig. 10 in Lawrence, 1990);

(35) gives $s^2(x_V)=4(1-\delta)/(3+\kappa_V)$;

we have $2v_1^2(x_V)=\delta\xi_1(x_V)$ and $2v_2^2(x_V)=\delta\xi_2(x_V)$ because $F_1(x_V)=F_2(x_V)=\delta$;

from (33)-(35) one gets equation $s(s^2-(1-\delta))=(2\delta-1)(1-\delta)A/\underline{b}$ which shows that s(x) takes maximal value $\sqrt{1-\delta}$ at $x=\pm\infty$ and has the minimum $s^2(0)=4(1-\delta)/(3+\kappa_V)$ at x=0, therefore $x_V=0$. Putting x=0 in the equation for s gives $A^2=4\kappa_V(1+\kappa_V)/(3+\kappa_V)^3$;

from (27) we have $q_1^2=4\delta/(3+\kappa_v)^3$, $q_2^2=4\delta/(1+3/\kappa_v)^3$.

Thus $x_V>0$ if $\delta<1/(1+\sqrt{1-\varepsilon})$ and $x_V<0$ if $\delta>1/(1+\sqrt{1-\varepsilon})$; $q_1>q_2$ and $\xi_1(x_V)>\xi_2(x_V)$ for $\delta=1/(1+\sqrt{1-\varepsilon})$.

Appendix D. **General solution**

Introducing the parameter $\underline{x}=\tanh^{-1}(2\xi_2-1)$ the solution of the system (31)-(35) can be written as

$$-\infty<\underline{x}<\tanh^{-1}(1-2\varepsilon\delta)$$

$$2\xi_1=1-\tanh\underline{x}-\varepsilon\delta^2(1+\tanh\underline{x})^2/(2(1-\varepsilon\delta)s^2),$$

$$2\xi_2=1+\tanh\underline{x},$$

$$2v_1=\delta(1+\tanh\underline{x})/(s\sqrt{1-\varepsilon\delta}),$$

$$2v_2=(1-\delta)(1-\tanh\underline{x}-\varepsilon\delta^2(1+\tanh\underline{x})^2/(2(1-\varepsilon\delta)s^2))/s,$$

$$4s^2=1+(2\delta-1)\tanh\underline{x}+((1+(2\delta-1)\tanh\underline{x})^2-4\varepsilon\delta^2(1-\delta)(1+\tanh\underline{x})^2/(1-\varepsilon\delta))^{1/2}$$

$$\underline{b}(x)=4As/(\cosh^{-2}\underline{x}+\varepsilon\delta^2(1+\tanh\underline{x})^3/(2(1-\varepsilon\delta)s^2)).$$

Here $A(\varepsilon,\delta)$ is given by (38)-(40). The analytical formulas for graphs presented on Fig. 9(b) (with ε=.5 and $\delta\approx$ .8) in Lawrence (1990) are given above.

Appendix E. Notation

The following symbols are used in sections 1-3 of this paper:

$A(\varepsilon,\delta)$ = is introduced by (27), the analytical expression is given by (38)-(40);

b(x) = the channel width;

b_0 = the minimal channel width;

$\underline{b}(x)=b(x)/b_0$ = the non-dimensional channel width;

$c=\sqrt{2\varepsilon gH}$ = the maximal velocity of an exchange flow;

g = the gravitational acceleration;

H_1 and H_2 = the lighter and denser fluid levels far from a contraction;

H = max (H_1,H_2);

Q = the discharge;

$q=Q/(cb_0H)$ = the discharge coefficient or non- dimensional discharge;

$s^2(x)=(1-\delta)\xi_1+\delta\xi_2$, see (35);
u = the velocity;
v=u/c = the non-dimensional velocity;
x = the long channel coordinate;
x_* = the position of the plunge point;
$\underline{x}$ = the parameter ($-\infty < \underline{x} < \infty$) in (47)-(52);
α = the parameter ($0< \alpha < \infty$) in (38)-(40);
ρ = fluid density;
$\varepsilon=(\rho_2-\rho_1)/\rho_2$ = the positive relative density difference;
$\gamma=(H_1-H_2)/H$ = the relative reservoir level difference;
η = thickness of the layer;
$\xi=\zeta/\varepsilon H$ = the non-dimensional free surface displacement;
$\xi_1=\eta_1/H$ and $\xi_2=\eta_2/H$ are the non-dimensional thickness of the lighter and denser layers;
$\delta=\gamma/\varepsilon$ = the ratio of relative reservoir level difference to relative density difference;
$\kappa(x)=\xi_2(x)/\xi_1(x)$ = the ratio of the layer thicknesses.

Subscripts
0 = at the narrowest cross-section;
1 = layer of lighter fluid;
2 = layer of denser fluid.

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LIST OF FIGURES

Figure 1. Plan view (a) for flow through a contraction and side views (b)-(h) for various flow regimes. Calculations of free surface (solid lines) and interface (dashed lines) were made for $\varepsilon=10^{-3}$ using (1) and (2) for (b)-(d), (33) and (34) for (e) and (7)-(10) for (f)-(h). The corresponding values of the parameter δ are (b) δ = -667; (c) δ = -62.5; (d) δ =-0.25; (e) δ = 0.25; (f) δ = 1.1; (g) δ = 62.5 and (h) δ = 667.

Figure 2. Graphs of q_1^2 and q_2^2 against δ for ε = 0, 0.25, 0.5, 0.75, 1.

Figure 3. Discharge coefficients q_1 and q_2 against δ for ε=0 (———), 0.25 (– – –), 0.4 (.......), 0.6 (– . – . – .).

Figure 4. Non-dimensional layer thickness $\xi_1(0)$, $\xi_2(0)$ and free surface displacement $\zeta(0)/H$ against δ for $\varepsilon=0$ (– . – . – .), 0.25 (-------), 0.4 (———), 0.6 (– – –).

Figure 5. Non-dimensional layer thickness $\xi_1(0)$ and $\xi_2(0)$ against δ for $\varepsilon=0$ (———), 0.25 (– . – . – .), 0.4 (———), 0.6 (.......) and Wood's (1970) experimental data. The numbers in the parentheses show the values of ε for corresponding experiment.