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# Steady flow between two reservoirs of fluid with different densities and levels through a rectangular channel section of varying depth and width

A. Odulo \*, J.C. Swanson

Applied Science Associates, 70 Dean Knauss Drive Narragansett, RI 02882, USA Received 15 November 1996; revised 19 December 1997; accepted 27 January 1998

### Abstract

The steady hydrostatic flow through a channel of rectangular cross section connecting reservoirs of infinite width and depth and containing inviscid fluids of different densities and levels is studied. The main goal is the determination of the discharges of the lighter and denser fluids in terms of the external conditions (reservoir levels, fluid densities and variation of width and depth along a channel). It is shown that the key parameter is  $\delta$ , which is the ratio of relative reservoir level difference,  $\gamma$ , to relative density difference,  $\varepsilon$ . If  $\delta < 0$  then the denser fluid plunges under the stationary lighter layer. If  $\delta > \delta_*$  (1 <  $\delta_*$  < 1.5) then the lighter fluid runs up on a wedge of stationary heavier fluid. Here  $\delta_*$  depends on the geometry of the constriction. The solutions describing these regimes are stated. If  $0 < \delta < \delta_*$  then both layers are in motion. A qualitative analysis of the solution for arbitrary bottom shape and channel width and arbitrary  $\varepsilon$  is presented and the problem is reduced to a system of two equations which can be easily solved numerically for any particular channel profile. We give detailed analyses for the following two cases: 1) the narrowest width of the channel is on the side of the heavier fluid and the top of the sill is on the side of lighter fluid; 2) the minima in channel depth and width coincide. In the second case the discharges for one class of geometries in the Boussinesq approximation are calculated and discussed. © 1998 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

Many deep estuaries and semi-enclosed seas are separated from the open ocean by constrictions. The exchange through the constrictions affect the hydrography of such

<sup>\*</sup> Corresponding author.

estuaries and seas and is important to the ocean and its adjoining shelf seas as well (see, e.g., Hogg and Huang, 1995). Often flows through the constrictions change from an exchange regime to one layer flow in either direction. Not only is the instantaneous 'net discharge' not zero most of the time, but the 'net discharge' averaged over time is not zero as well (due to river discharge or evaporation, for example). Therefore, it is important to be able to calculate the discharges of straits and silled fjords for all possible regimes and with arbitrarily changing width b(x) and depth h(x) along the channel.

In the cases of an exchange flow through a pure contraction or a pure sill, the layer discharges do not depend on depth or width profiles as long as the minimal cross section stays the same. However, the discharges for a contraction are different from those for a sill. In this paper we are studying the effect of the change in depth vs. width along a channel on the layer discharges. There are dozens of papers studying the effects of friction, stratification, rotation, nonrectangular cross section and time dependence on a flow through a constriction with constant width or depth. These and other effects are outside the scope of our consideration.

In this paper we consider a steady flow through a constriction between two large basins, where each basin contains a homogeneous fluid of different density and surface level. Such a flow, called 'a lock exchange flow' (Wood, 1970), differs from the outflow through a channel from a large basin of two layered fluid, called 'withdrawal from a layered fluid' (Wood, 1968) when both layers flow freely in the same direction. Both type of flow are described by the same equations (Bernoulli and continuity) but with different conditions far from the constriction. In the 'withdrawal' case, the thicknesses of the layers diminish in the same direction. In contrast to a flow of a two layered fluid in a channel with an obstacle, a contraction or both (see examples in Baines (1995) (Chap. 3), we consider the limit when the depth and width of the minimal cross section of a constriction are much smaller then the width and depth of the basins. In this limit we can assume that the width and the depth of the basins infinitely increase on both sides of a sill and a narrow, respectively, where they have unique minima (see Fig. 1).

In the case of the free outflow (see, e.g., Bakhmeteff (1932), (p. 41) or Henderson (1966), Section 6.4) of a homogeneous fluid from a large basin through an open channel of rectangular cross section the discharge can be easily found using the specific energy equation, introduced by Bakhmeteff (1932) (Section 15), see also Henderson (1966), (p. 31). In this case the *critical depth* (Bakhmeteff (1932), p. 35) will be at  $x = x_+$  so that the discharge  $Q = 2c_0(1 + h(x_+)/H)^{3/2} Hb(x_+)/\sqrt{27}$ . Here, H is the reservoir level above the sill datum, h(x) is the channel depth calculated from sill crest,  $c_0 = (2 gH)^{1/2}$  is the velocity reached by an initially motionless fluid parcel falling from a height H and  $x_+$  is the position of the minimum of the function

$$S(x) = (H + h(x))b^{2/3}(x).$$
 (1)

The function S(x) has a minimum between the sill and the narrow and increases monotonically to infinity with |x| outside this range. Between the sill and the narrow S(x) can have complicated behavior and we assume that S(x) has a unique minimum for simplicity. The discharge depends on the minimum of the function S(x) but will not change with any changes of b(x) or h(x) as long as this minimum stays the same. The discharge of a fluid of density  $\rho_2$ , completely covered by motionless fluid of density  $\rho_1$ , outflowing from a large basin is  $Q = 2c(1 + h(x_+)/H)^{3/2}Hb(x_+)\sqrt{27}$ . Here  $c = \sqrt{2\varepsilon gH}$  (which is less than  $c_0$  due to Archimedes force) is the velocity reached by an initially motionless fluid parcel falling from a height H in the fluid of density  $\rho_1$ ,  $\varepsilon = (\rho_2 - \rho_1)/\rho_2$ . If the minima in channel depth and width coincide, then  $Q = 2Q_0/\sqrt{27}$ , where  $Q_0 = cb_0H$ ,  $b_0 = \min b(x)$ . We will use c and  $Q_0$  as a typical velocity and a typical discharge to introduce the nondimensional variables.

The exchange flow case, however, is not quite so straightforward. In addition to the Bernoulli and continuity equations we need two more equations to find the unknown layer discharges. Wood (1970) showed that the condition that the thicknesses of the moving layers decrease smoothly from their upstream values to zero in the infinitely wide downstream reservoirs, gives these two additional equations which provide the complete system of algebraic equations determining a unique solution. This system relates the layer discharges, two unknown locations (so-called critical points) and the layer thicknesses at these locations. Farmer and Armi (1986) and Dalziel (1991) considered the geometries for which one or both critical points are known a priori. Some authors assumed that the critical points are at the sill and at the narrowest section (see, e.g., Bryden and Kinder (1991), Eqs. (12)–(13); Oguz et al. (1990), Eq. (16)). But it is obvious that this never occurs if the width changes at the sill and the depth changes at the narrow (see Eq. (13a) in Armi (1986)).

The results of numerical calculations with the Boussinesq approximation for channels of rectangular cross section, where the width changes in the region with flat bottom and the depth changes with constant width, were presented by Farmer and Armi (1986) (depth increasing to infinity on both sides of a sill in a channel of uniform width, then width increasing to infinity in infinitely deep reservoirs) and Dalziel (1991) (the same as above except with the width monotonically increasing in the reservoirs with a constant finite depth). They used the difference of the nondimensional discharges as the independent parameter and showed graphs of velocities and thicknesses of the layers at the sill and nondimensional discharges vs. this parameter.

In this paper a steady flow through a channel (with rectangular cross section and varying width b(x) and depth h(x) along the channel) connecting reservoirs with fluids of different densities,  $\rho_1$  and  $\rho_2$ , and levels,  $H_1$  and  $H_2$ , is examined (see Fig. 1). To avoid unnecessary complications we assume that the channel width b(x) has a unique minimum  $b_0$  at  $x = x_h$  and monotonically increases to infinity at  $x \to \pm \alpha$ , the channel depth h(x) has a unique minimum equal to zero at x = 0 and monotonically increases to infinity at  $x \to \pm \alpha$  (so b(x) and h(x) are **arbitrary** functions under these conditions). The key parameter is  $\delta = \gamma/\varepsilon$ , which is the ratio of relative reservoir level difference  $\gamma = \zeta_{\infty}/H$  to relative density difference  $\varepsilon$ . Here  $\zeta_{\infty} = H_1 - H_2$ ,  $H = \max(H_1, H_2)$ . If  $\delta < 0$  then the denser fluid plunges under a motionless lighter layer (regimes 1–3, Fig. 1c-e; the discharge of lighter fluid  $Q_1 = 0$ . If  $\delta > \delta_*$  (by definition,  $\delta_*$  is the minimum  $\delta$  when the denser fluid is arrested) then the lighter fluid runs up over a motionless wedge of denser fluid (regimes 5-7, Fig. 1g-i; the discharge of denser fluid  $Q_2 = 0$ ). The solutions for regimes 1–3 and 4–7 are similar to well-known solutions for one-layer flow. Therefore, the results for these regimes are simply stated in Section 2 and their derivation is described. Regimes 1, 2, 3, 5, 6 and 7 differ in the positions of the tip of the wedge of the stagnant fluid (the plunge point; see Wood, 1970, p. 676) and in the dependence of  $Q_1$  and  $Q_2$  on  $\delta$ .

If  $0 < \delta < \delta_*$  then both layers are in motion (regime 4, Fig. 1f). Unlike the cases of pure contraction or pure sill in this case the discharge coefficients  $q_1(\varepsilon, \delta)$  and  $q_2(\varepsilon, \delta)$  (which are nondimensional discharges  $q_{1,2} = Q_{1,2}/Q_0$ ) are different for channels with different h(b). As for a pure contraction (Wood, 1970), for a channel with arbitrary width b(x) and depth h(x) the requirement that the thicknesses of both layers monotonically decrease in the respective directions of their flows gives the complete system of eight equations which determine a unique solution. This system is reduced to two equations in Appendix A. We analyze two cases in detail: 1) the narrowest width  $b_0 = b(x_b)$  of the channel is on the side of the heavier fluid ( $x_b > 0$ ) and the top of the sill (at x = 0) is on the side of the lighter fluid; 2) the minimum channel depth and width are at the same location, x = 0.

In the first case, when  $\delta$  increases from 0 (with  $q_1 = 0$ ) to  $\delta_*$  (with  $q_2 = 0$ ) the position of one critical point decreases from  $x_+$  to  $x_* < 0$  and the position of another critical point decreases from  $\infty$  to  $x_b$ . The simple equations from which one can find  $\delta_*$  and  $x_*$  are presented. In the Boussinesq approximation ( $\varepsilon \ll 1$ ) the problem is reduced



Fig. 1. Top (a) and side (b) views of a channel geometry. The origin of the along channel axis x is chosen at the top of a sill. Side views for the seven possible flow regimes are shown in (c)-(i).



Fig. 1 (continued).



Fig. 1 (continued).

to one equation which connects the critical point positions (compare with steps (i)–(xi) of Dalziel (1991), p. 147). Having the critical points one can calculate corresponding values of  $\delta$  and the discharge coefficients from explicit algebraic formulas.

In the second case one critical point can be taken at the narrowest section. Then another critical point varies over the interval  $[\infty, x_*]$  when  $\delta$  varies over the interval  $[0, \delta_*]$ . Here,  $x_*$  is the furthest extent of the tip of the wedge of the stagnant denser fluid. The problem is reduced to one equation. The calculations are made for the particular case when the bottom profile h(x) and the channel width b(x) are connected by the relation  $b^2(x) = b_0^2 [1 + a^2 h(x)/H]^3$ . Here, h(x) is an arbitrary positive function with h(0) = 0; a,  $b_0$  and H are arbitrary constants. In this case the discharge coefficients  $q_1(\varepsilon, \delta, a)$  and  $q_2(\varepsilon, \delta, a)$  do not depend on h(x). It is shown how the flow changes when  $a^2$  increases from 0 (sill) to  $\infty$  (contraction).

A discussion of the relative effects of the changes of the depth and channel width is included in Section 4.

The conventions adopted throughout this paper are as follows:

subscript x means a derivative; subscripts -, +, \*, b, c,  $\eta$  or u indicate the value of a corresponding function at the point  $x_-$ ,  $x_+$ ,  $x_*$ ,  $x_b$ ,  $x_c$ ,  $x_\eta$  or  $x_u$ , respectively; the lighter fluid moves from left to right and the denser fluid moves from right to left; all values and parameters are positive except the levels  $H_1$  and  $H_2$  of the lighter and denser fluids above the sill, their difference  $\zeta_{\infty}$  and the parameters  $\gamma$  and  $\delta$ .

### 2. Plunging and run up

We consider the problem sketched in Fig. 1 of two-layer flow through a channel of rectangular cross section with slowly varying width and depth connecting two reservoirs of infinite width and depth. We assume that the bottom topography h(x) monotonically decreases  $(h_x < 0)$  at x < 0 and increases  $(h_x > 0)$  at x > 0,  $h_x(0) = 0$ ; the channel width varies so that  $b_x < 0$  at  $x < x_b$  and  $b_x > 0$  at  $x > x_b$ ,  $b_x(x_b) = 0$ . We also assume that width and depth change slowly (i.e.,  $h_x \ll 1$  and  $b_x \ll 1$ ) in order to use the hydrostatic approximation and to neglect the dependence on a second horizontal coordinate. We limit our discussion to the steady flow of two homogeneous inviscid fluids. It is obvious that parts of the channel where both width and depth are constants can be excluded from consideration without any effect on the result, so long as we neglect mixing and friction.

We introduce the following non-dimensional variables

$$\xi = \zeta/\varepsilon H, \ \xi_1 = \eta_1/H, \ \xi_2 = \eta_2/H, \ v_1 = u_1/c,$$
  

$$v_2 = u_2/c, \ h_1 = \xi_1 \underline{b}^{2/3}, \ h_2 = \xi_2 \underline{b}^{2/3}$$
  

$$\underline{h} = h/H, \ \underline{b} = b/b_0, \ \varphi(x) = S(x)/Hb_0^{2/3},$$
(2a)

and non-dimensional coefficients

$$q_1 = Q_1 / Q_o, q_2 = Q_2 / Q_o.$$
 (2b)

Here  $\zeta$  is free surface displacement, and  $\xi_1$ ,  $\xi_2$ ,  $v_1$ ,  $v_2$ ,  $Q_1$  and  $Q_2$  are the thickness, velocity and discharge of the lighter and denser fluid, respectively. Fig. 1 shows dimensional variables.

#### 2.1. A denser fluid plunging under a stationary lighter layer

If the level of the heavier fluid reservoir is higher  $(H_2 \ge H_1)$ , then the lighter fluid is at rest (velocity  $u_1 = 0$ ; Fig. 1c–e). The position where the flowing layer plunges under the stationary layer (Wood, 1970, p. 676) is denoted as  $x_*$ . From the Bernoulli and

continuity equations of the moving denser layer one can get the non-dimensional specific energy equation (Bakhmeteff (1932), Section 1.5 or Henderson (1966), Section 2.6)

$$h_2 + \varepsilon q_2^2 / h_2^2 = \varphi(x) \quad \text{for } x \ge x_*,$$
(3a)

and

$$h_2 + q_2^2 / h_2^2 = \mu(x) \text{ for } x_* \ge x,$$
 (3b)

here

$$\mu(x) = \left(1 + \underline{h}(x) - \delta(1 - \varepsilon)\right)\underline{b}^{2/3}(x).$$
(4)

We denote the position of the minimum of the function  $\mu(x)$  as  $x_{-}$ , which is the function of  $\delta$  and  $\varepsilon$ ,  $x_{-}(\delta,\varepsilon)$ . When  $\delta = 0$  then  $x_{-}$  and  $x_{+}$  coincide,  $x_{-}(0,\varepsilon) = x_{+}$ .

The specific energy curve (see Fig. 30 in Bakhmeteff (1932) or Figs. 2–3 in Henderson (1966)), defined by the left side of the Eqs. (3a) and (3b) as function of  $h_2$ , tends to infinity as  $h_2 \rightarrow 0$  and  $h_2 \rightarrow \infty$ . The curve has a minimum, corresponding to a certain position called the critical point and designated as  $x_c$ . There are three possibilities: the critical point is upstream ( $x_c > x_*$ , regime 1,  $x_c = x_+$ ), at ( $x_c = x_*$ , regime 2) or downstream ( $x_c < x_*$ , regime 3,  $x_c = x_-$ ) of the plunging point (see Fig. 1c–e).

For regime 1 the condition that the left side of the Eq. (3a) must have a minimum at  $x = x_+$ , where the right side has a minimum, gives  $\varepsilon q_2^2 = 4\varphi_+^3/27$ . For regime 3 the condition that the left side of the Eq. (3b) must have a minimum at  $x = x_-$ , where the



Fig. 2. Typical behavior of channel geometry functions  $\varphi_x$  and  $b_x^{2/3}$  along a channel with  $h_x$  and  $b_x$  being monotonically increasing and the position of minimum width  $x_b > 0$ . Behavior of the 'kinetic flow factor'  $F_2(x;\delta,\varepsilon) = 2q_2^2(\delta,\varepsilon)/h_2^3(x;\delta,\varepsilon)$  for some particular  $\varepsilon$  and  $\delta$  inside the interval  $[0, \delta_*]$  is also shown.





Fig. 3. The intersections of the functions  $F_2 \varphi_x$  and  $\delta b_x^{2/3}$  give the graphical solution of Eq. (31).

right side has a minimum, gives  $q_2^2 = 4\mu_-^3/27$ . For regime 2 putting  $x = x_*$  and  $h_2(x_*) = \varphi(x_*) + \varepsilon \delta b_{2/3}(x_*)$  into Eq. (3a) we get  $q_2^2 = -\delta h_2(x_*)^2 b_{2/3}(x_*)$ . Putting  $h_2(x_*) = (h(x_*) + 1 + \varepsilon \delta) b_{2/3}(x_*)$  into Eq. (3a) we get the equation for the position of the plunge point

$$b(x_*)(1+\varepsilon\delta+h(x_*)) = q_2/\sqrt{-\delta}, \qquad (5)$$

for regime 1,  $x_* < x_+$ , and for regime 3,  $x_* > x_-$ . It was pointed out by Wood (1970), (pp. 677–8) that  $h_{2x}$  is discontinuous at the plunge point but only the condition that  $h_2$  is a continuous function is used.

The regime boundaries in terms of  $\varepsilon$  and  $\delta$  can be found from the conditions  $\xi_2(x_*) < \xi_2(x_c)$  for regime 1 and that  $\xi_2(x_c) < \xi_2(x_*)$  for regime 3 and are presented in the first column of the Table 1. The first three rows of Table 1 present the discharge coefficients  $q_1$  and  $q_2$  and the nondimensional thicknesses of lighter and denser layers  $\xi_{1c} = \eta_1(x_c)/H$  and  $\xi_{2c} = \eta_2(x_c)/H$  at the position  $x_c$  as functions of  $\varepsilon$  and  $\delta$  for these three regimes.

#### 2.2. A lighter fluid running up over a stationary denser wedge

If the level of the lighter fluid reservoir is so high that  $\delta \ge \delta_*$   $(H_1 \ge H_2 + \varepsilon \delta_* H_1)$ , then the denser fluid is at rest,  $u_2 = 0$ , and the lighter fluid runs up over a wedge of the denser fluid (Fig. 1g-i).

From the Bernoulli and continuity equations of the moving lighter layer one can get the non-dimensional specific energy equation

$$h_1 + \varepsilon q_1^2 / h_1^2 = \varphi(x) \quad \text{for } x_* \ge x, \tag{6a}$$

and

$$h_1 + q_1^2 / h_1^2 = \delta \underline{b}^{2/3}(x) \quad \text{for } x \ge x_*.$$
 (6b)

	$q_1$	$q_2$	$\xi_{2c}$	$\xi_{1c}$
Regime 1: $-\delta \ge (1 + h(x_+))/3\varepsilon$	0	$2 arphi_+^{3/2}/\sqrt{27arepsilon}$	$2(1+h_+)/3, x_c = x$	_
Regime 2: $(1 + h(x_+))/3\varepsilon \ge -\delta \ge$				
$(1+h(x))/(2+\varepsilon)$	0	$\underline{b}_*  \xi_{2c*} \sqrt{-\delta}$	$1 + h_* + \varepsilon \delta, \ x_c = x *$	0
Regime 3: $(1 + h(x))/(2 + \varepsilon) \ge -\delta \ge 0$	0	$2\mu_{-}^{3/2}/\sqrt{27}$	$2(1+h_{-}-\delta(1-\varepsilon))/3, x_{c}=x_{-}$	$(1 + h_{-} + \delta(2 + \varepsilon))/3, x_{c} = x_{-}$
Regime 5: $\delta_* \le \delta \le 3(1+h_+)/(2+\varepsilon)$	$2(\delta/3)^{3/2}$	0	$1 + h_{\rm b} - \delta(2 + \varepsilon)/3, \ x_c = x_b$	$2\delta/3, x_c = x_b$
Regime 6: $3(1+h_+)/(2+\varepsilon) \le \delta \le$	$b_*(1+h_*-\varepsilon\delta)$	0	0	$(1+h(x_+)-\varepsilon\delta)/(1-\varepsilon), x_c=x_*$
$(1+2\varepsilon)(1+h(x_+))/3\varepsilon$	$(\delta - 1 - h_{-})^{1/2}/(1 - \varepsilon)^{3/2}$			
Regime 7: $(1+2\varepsilon)(1+h(x_+))/3\varepsilon \le \delta$	$2 \varphi_+^{3/2} / \sqrt{27\varepsilon}$	0	_	$2(1+h_+)/3, x_c = x_+$

As in the case with motionless lighter fluid, three regimes are possible: the critical position lies downstream ( $x_c > x_*$ , regime 5,  $x_c = x_b$ ), at ( $x_c = x_*$ , regime 6) or upstream ( $x_c < x_*$ , regime 7,  $x_c = x_+$ ) of the plunging point (Fig. 1g–i).

In particular for regime 5 we get  $2q_1^2 = \xi_{1c}^3 = (2\delta/3)^3$ . The position of the plunging point  $x_*(\varepsilon, \delta)$  can be found from the equation

$$4(1-\varepsilon)^2\delta^3 = 27(1+\underline{h}_*-\varepsilon\delta)^2(\delta-1-\underline{h}_*)\underline{b}_*^2.$$
<sup>(7)</sup>

Then one gets the moving layer thickness at the position of the plunging point from

$$(1-\varepsilon)\xi_{1*} = 1 + \underline{h}_* - \varepsilon\delta.$$
(8)

The upper regime boundary can be found from the condition  $\xi_{1*} \ge \xi_{1c}$ .

It is obvious that the depth profile under the motionless denser fluid layer does not have any influence on the lighter fluid motion. Therefore we would expect that, for regime 5, the discharge  $Q_1^2 = 8 \zeta_{\infty}^3 g b_0^2 / (27\varepsilon^2)$  and the thickness of the lighter fluid layer  $\eta_1$  at  $x \ge x_*$  are the same as for a contraction alone. If we take  $b(x_c)$  and  $H + h(x_c)$  instead of  $b_0$  and H in Eq. (2b) then for regimes 6 and 7 the discharge coefficient  $q_1$  will be identical to that for pure contraction.

The last three rows of Table 1 present  $\xi_{1c}$  and  $\xi_{2c}$  and the discharge coefficients  $q_1$  and  $q_2$  as functions of  $\varepsilon$  and  $\delta$  for regimes 5–7.

#### 3. Two-layer exchange flow

The solution for regime 4 will be obtained in this section from the Bernoulli and continuity equations and the requirement that the thickness of the layers  $\eta_1$  and  $\eta_2$  continuously decrease from their maximum values to 0.

## 3.1. Arbitrary channel geometry

If the level of the lighter fluid is only slightly higher than the level of the heavier fluid, so that  $0 < \delta < \delta_*$ , then two-directional exchange flow occurs. The Bernoulli and continuity equations for the upper layer are

$$u_1^2 = 2g\zeta,\tag{9}$$

$$u_1\eta_1 b = Q_1,\tag{10}$$

and for the lower layer are

$$u_2^2 = 2g(\zeta - \zeta_\infty + \varepsilon \eta_1), \tag{11}$$

$$u_2\eta_2 b = Q_2. \tag{12}$$

We also have from the definition of the free surface displacement  $\zeta$  that

$$\eta_1 + \eta_2 + \zeta = H_1 + h(x). \tag{13}$$

To obtain Eqs. (9) and (11) we have used the boundary conditions

$$u_1 \to 0, \zeta \to 0 \quad \text{at } x \to -\infty,$$
 (14)

$$u_2 \to 0, \, \zeta \to \zeta_{\infty}, \, \eta_1 \to 0 \quad \text{at } x \to \infty.$$
 (15)

We already have the solution for  $\delta = 0$  and  $\delta = \delta_*$  (see regimes 3 and 5 in Table 1). Even before solving the exchange problem we see that when  $\delta$  increases from 0 to  $\delta_*$  then  $q_1$  increases from 0 to  $2\delta_*^{3/2}/\sqrt{27}$  and  $q_2$  decreases from  $2\varphi_+^{3/2}/\sqrt{27}$  to 0. For some purposes it can be sufficient to take  $q_1$  and  $q_2$  simply as linear functions of  $\delta$  for the exchange regime. It will be shown that to find the exact solution one must solve the system of two algebraic equations, which contain h(x), b(x) and their derivatives.

Using non-dimensional variables (introduced in Section 2) one can rewrite the system Eqs. (9)–(13) in the following form

$$v_1^2 = \xi, \tag{16}$$

$$v_1 \xi_1 \underline{b} = q_1, \tag{17}$$

$$v_2^2 = \xi - \delta + \xi_1, \tag{18}$$

$$v_2 \xi_2 \underline{b} = q_2, \tag{19}$$

$$\xi_1 + \xi_2 + \varepsilon \xi = 1 + h. \tag{20}$$

For a given channel geometry b(x) and h(x), density difference ratio  $\varepsilon$  and lighter and denser reservoir levels  $H_1$  and  $H_2$  ( $\delta = \zeta_{\infty}/\varepsilon H$ ,  $\zeta_{\infty} = H_1 - H_2$ ), the system of five equations Eqs. (16)–(20) contains five unknown functions  $\xi(x)$ ,  $\xi_1(x)$ ,  $\xi_2(x)$ ,  $v_1(x)$ and  $v_2(x)$  and two unknown discharge coefficients  $q_1$  and  $q_2$ . The discharge coefficients  $q_1$  and  $q_2$  can be found using the condition that  $\xi_{1x}$  and  $\xi_{2x}$  are bounded functions.

From Eq. (20) and the boundary conditions we have

$$0 \le \xi_1 \le \xi_1(-\infty) = 1 + \underline{h}(-\infty) \text{ and } 0 \le \xi_2 \le \xi_2(\infty) = (1 - \varepsilon\delta)(1 + \underline{h}(\infty)).$$

It follows from Eqs. (16) and (18) that  $v_1 = v_2$  at the point  $x = x_u$  where  $\xi_1 = \delta$  and

$$0 \le v_1 \le \sqrt{\delta} = v_1(\infty), 0 \le v_2 \le \infty.$$

In the Boussinesq approximation ( $\varepsilon = 0$ ) we have from Eqs. (17), (19) and (20)

$$q_r = q_2/q_1 = (1-\delta)/\delta + h(x_u)/\delta.$$

Thus,  $q_2 > q_1$  for  $\delta < 1/2$ . For a contraction alone  $h \equiv 0$  so  $q_r = (1 - \delta)/\delta$ . Therefore, for given  $\delta$ , the value of  $q_r(\delta)$  for a contraction alone is always smaller than for any other geometry.

The parameter  $\varepsilon$  is present only in Eq. (20). Therefore, the Boussinesq approximation ( $\varepsilon = 0$ ) and the rigid lid approximation ( $\xi_1 + \xi_2 = 1 + \underline{h}$ ) are equivalent for exchange flow.

Eliminating  $v_1$ ,  $v_2$  and  $\xi$  from the system Eqs. (16)–(20) one gets

$$h_1 + h_2 + \varepsilon q_1^2 / h_1^2 = \varphi(x)$$
(21)

$$\delta \underline{b}^{2/3}(x) + q_2^2 / h_2^2 = h_1 + q_1^2 / h_1^2$$
(22)

One can call Eq. (22) the specific energy equation of an exchange flow. For  $q_2 = 0$  Eq. (22) gives Eq. (6b). The interpretation of the Eq. (21) is geometrical and is especially clear in the Boussinesq approximation when this equation takes the form  $h_1 + h_2 = \varphi(x)$ .

We are looking for a continuous solution of the system Eqs. (21) and (22) with conditions

$$h_1 \to \infty, \quad h_2 \to 0 \quad \text{at } x \to -\infty,$$
 (23)

$$h_1 \to 0, \quad h_2 \to \infty \quad \text{at } x \to \infty.$$
 (24)

It is readily seen from Eqs. (21)–(24) that

$$(1-\varepsilon\delta)^{1/2}(1-\delta)q_1=\delta q_2,$$

in the case of a contraction alone  $(h \equiv 0)$  and

$$2q_1^2 = (2\delta/3)^3$$

in the case of a sill alone  $(b \equiv b_0)$ . Then from the condition that  $h_1$  continuously decreases from  $\infty$  to 0 one gets analytical expressions for  $q_1(\varepsilon, \delta)$  and  $q_2(\varepsilon, \delta)$  which do not depend on the behavior of b(x) or h(x). When both width and depth vary along a channel, it is more difficult to find  $q_1(\varepsilon, \delta)$  and  $q_2(\varepsilon, \delta)$  such that the solution of the system Eqs. (21) and (22),  $h_1(x)$  and  $h_2(x)$ , satisfies the conditions Eqs. (23) and (24). The rest of the paper deals with this problem.

From Eqs. (21) and (22) one can find (compare with (10c) and (10d) in Armi (1986))

$$D(x)h_{1x} = -R(x) \tag{25}$$

and

$$D(x)h_{2x} = P(x). \tag{26}$$

Here

$$D(x) = F_1 + F_2 - \varepsilon F_1 F_2 - 1, \qquad (27)$$

$$R(x) = \delta(\underline{b}^{2/3})_x - F_2 \varphi_x, \qquad (28)$$

$$P(x) = (1 - \varepsilon F_1) R(x) - D(x) \varphi_x, \qquad (29)$$

and

$$F_j(x) = 2q_j^2 / h_j^3(x) (j = 1, 2)$$
(30)

is the 'kinetic flow factor' which is ''twice the ratio of the kinetic energy head to the potential energy head'' (Bakhmeteff, 1932, p. 64). As x increases from  $-\infty$  to  $\infty$ ,  $F_1(x)$  increases from 0 to  $\infty$  and  $F_2(x)$  diminishes from  $\infty$  to 0. Recent authors used the notation  $F = Fr^2$ , where Fr is called the local Froude number (Baines (1995), p. 38 and Section 1.4). Fig. 2 shows the qualitative behavior of the functions  $F_2(x)$ ,  $\varphi_x(x)$  and  $(b_{2/3})_x$  in the case  $x_b > 0$ . One can see that for  $\delta > 0$  the function R(x) is positive as  $x \to \pm \infty$  but  $R(x_b) < 0$  and  $R(x_+) < 0$ . Thus the function R(x) has at least two zeros

$$R(x_1) = R(x_2) = 0.$$
(31)

For simplicity suppose that the equation R(x) = 0 has exactly two roots  $x_1$  and  $x_2$  (Fig. 3). In this case one can see from Eq. (25) that D(x) < 0 in the interval  $(x_1, x_2)$ . It is clear that

$$x_1 \le x_+ < x_b \le x_2. \tag{32}$$

The requirement that  $h_{1x}$  is a bounded function leads to the condition

$$D(x_1) = D(x_2) = 0.$$
(33)

Eqs. (31) and (33) correspond 'the regularity conditions' (13a) and (13c) in Armi (1986). Putting  $x = x_1$  and  $x = x_2$  into Eqs. (21) and (22) we have together with the four Eqs. (31) and (33) the system of eight equations for eight unknown values

$$x_i, q_1, q_2, h_{1i} = h_1(x_i) \text{ and } h_{2i} = h_2(x_i), \quad i = 1, 2.$$
 (34)

This system can be written in the form

$$\delta(\underline{b}^{2/3})_{xi} = F_{2i}\varphi_{xi},\tag{35}$$

$$F_{1i} + F_{2i} - \varepsilon F_{1i} F_{2i} = 1, (36)$$

$$\xi_{1i}(1+F_{1i}/2) = \delta + \xi_{2i}F_{2i}/2, \tag{37}$$

$$\xi_{1i}(1 + \varepsilon F_{1i}/2) + \xi_{2i} = 1 + \underline{h}(x_i), \tag{38}$$

where  $\xi_{1i} = h_{1i} / \underline{b}^{2/3}(x_i, \xi_{2i} = h_{2i} / \underline{b}^{2/3}(x_i)$  and

$$F_{1i} = 2q_1^2 / h_{1i}^3, \ F_{2i} = 2q_2^2 / h_{2i}^3.$$
(39)

Armi (1986) called the left side of Eq. (36) 'the composite Froude number'.

The system Eqs. (35)–(38) gives us eight equations for eight unknowns Eq. (34) and can be reduced to the system of two equations for  $x_1(\varepsilon, \delta)$  and  $x_2(\varepsilon, \delta)$  (see Eqs. (A6) and (A7) in Appendix A). In the Boussinesq approximation ( $\varepsilon = 0$ ) this system can be reduced to one equation which connects  $x_1$  and  $x_2$  (see Eq. (B1) in Appendix B). Some authors took  $x_1 = 0$  and  $x_2 = x_b$  and ignored two Eq. (35). In some particular cases such an approach gives the solution (e.g., the solution shown in Fig. 8 by Helfrich, 1995) which differs slightly from the exact one.

If the free surface is a horizontal plane ( $\delta = 0$ ) then  $x_2 = \infty$  and  $x_1 = x_+$ . The form of the interface in the case  $\delta = 0$  is the same as the form of the free surface for one layer flow. For  $\delta = 0$  we have

$$q_1 = 0, h_{12} = 0, F_{21} = 1, x_1 = x_+, x_2 = \infty, 2q_2^2 = h_{21}^3 = (2\varphi(x_+)/3)^3.$$
 (40)

It should be remembered that  $x = x_+$  is the location where the function  $\varphi(x)$  has a minimum.

If  $\delta = \delta_*$  we have

$$q_2 = 0, h_{21} = 0, F_{22} = 0, x_1 = x_*, x_2 = x_b, 2q_1^2 = h_{12}^3 = (2\delta_*/3)^3.$$
 (41)

The values  $x_*$  and  $\delta_*$  can be found from Eqs. (A10) and (A11), which in the Boussinesq approximation take the form

$$\frac{\underline{b}^{3}\underline{h}_{x}[\underline{b}(1+\underline{h})]_{x}^{2} = \left[2\underline{b}_{x}(1+\underline{h})/3 + \underline{b}\underline{h}_{x}\right]^{3}}{\left[\underline{b}^{2}(1+\underline{h})^{2}\right]_{x}\delta_{*}} = \left[\underline{b}^{2}(1+\underline{h})^{3}\right]_{x}.$$

For example, for geometry considered by Helfrich (1995), (p. 365)

$$\underline{b} = 4 - 3/\exp(\alpha^2(x-1)^2), \quad \underline{h} = \tanh^2 \beta x \quad \text{at } x < 0$$

one can find for  $\beta = 3.75$ 

and  $x_*$  is close to 0. For  $\alpha = 0.637$  and  $\beta = 0.01$  we get  $\delta_* \approx 1.01$  and  $x_* \approx -2.32$ .

Generally, when  $\delta$  increases from 0 to  $\delta_*$ ,  $x_1$  decreases from  $x_+ > 0$  to  $x_* < 0$  and  $x_2$  decreases from  $\infty$  to  $x_b > 0$ . The typical behavior of the functions  $x_1(\delta)$  and  $x_2(\delta)$  is shown in Fig. 4.



Fig. 4. The position of the critical points  $x_1$  and  $x_2$  vs. the parameter  $\delta$  for  $x_b > 0$ . At these points  $F_1 + F_2 - \varepsilon F_1 F_2 = 1$ .



Fig. 5. Graphs of maximum  $\delta$  for which the exchange regime exists,  $\delta_*$ , and upper layer thickness at the narrows  $h_{10*}$  for  $\delta = \delta_*$  vs.  $a^2$  for channel geometry given by Eq. (42).

# 3.2. The channel depth is a linear function of the channel width to the 2/3 power

Using the Boussinesq approximation ( $\varepsilon = 0$ ) let us consider a particular case when the channel depth is a linear function of the channel width to the 2/3 power (the general case when the minima in channel depth and width coincide is discussed in Appendix C)

$$\underline{h}(x) = \left(\underline{b}^{2/3}(x) - 1\right)/a^2 \tag{42}$$

Here b is an arbitrary function of x and a is an arbitrary constant. The specific energy Eq.  $(\overline{22})$  takes the form

$$a\delta\left(\sqrt{h_1 + h_2 + \lambda^2} - \lambda\right) + q_2^2/h_2^2 = h_1 + q_1^2/h_1^2$$
(43)

where  $2\lambda = a - 1/a$ . The channel geometry is represented in the specific energy Eq. (43) only by the parameter  $a^2$ . Therefore the discharge coefficients  $q_1$  and  $q_2$  depend on  $\delta$  and  $a^2$  but do not depend on a particular form of the channel width b(x). From Eqs. (35)–(39) one can get (see Eqs. (C5) and (C6))

$$2q_1^2 = h_{10}^3 (1 + 2\delta - 3h_{10}) = h_{12}^3 \Big( 1 - 0.5a\delta / \sqrt{h_{12} + h_{22} + \lambda^2} \Big), \tag{44}$$

$$2q_2^2 = (1 - h_{10})^3 (3h_{10} - 2\delta) = 0.5h_{22}^3 a\delta / \sqrt{h_{12} + h_{22} + \lambda^2}.$$
 (45)

These equations together with Eq. (37) written in the form

$$3h_{12} = 2a\delta \left[ -\lambda + \left( 1.25(h_{12} + h_{22}) + \lambda^2 \right) / \sqrt{h_{12} + h_{22} + \lambda^2} \right]$$
(46)

give us a complete system to find  $h_{10}(a,\delta)$ ,  $h_{12}(a,\delta)$  and  $h_{22}(a,\delta)$ . The system Eqs. (44)–(46) can be reduced to one equation

$$2\delta = 1.6[(\alpha^{3} - 1)/(\alpha^{3} - \beta^{3})]$$

$$\times \left[a\lambda + \alpha + \sqrt{(a\lambda + \alpha)^{2} + 1.25a^{2}(\lambda^{2} + \alpha\beta^{3})}\right]/a^{2}$$

$$= \left[(\alpha^{3} - 1)/(\alpha^{3} - \beta^{3})\right] \left[-\beta^{3} + (3\beta + 2.4\lambda^{2} + 0.6a\lambda\beta^{3})/(0.6(a\lambda + \alpha) - (\alpha^{3} - 1)/(\alpha^{2} + \alpha\beta + \beta^{2})]\right]$$
(47)



Fig. 6. Graphs of (a) upper layer thickness at the narrows  $\xi_{10}$ , (b) velocities of the lighter and denser fluids at the narrows  $v_{10}$  and  $v_{20}$  and (c) non-dimensional discharges  $q_1$  and  $q_2$  against  $\delta$  for channel geometry given by Eq. (42) with  $a^2 = 0$  (heavy solid lines),  $a^2 = 1$  (dashed lines) and  $a^2 = \infty$  (solid lines).

which connects  $\alpha = h_{12}/h_{10}$  and  $\beta = h_{22}/h_{20}$ . Varying  $\alpha$  from 0 to  $\alpha_*$  one can calculate  $\beta$  and  $\delta$  from Eq. (47),  $h_{10}$  from

$$(\alpha - \beta)h_{10} = (0.5a\delta(\alpha^3 - \beta^3)/(\alpha^3 - 1))^2 - \beta - \lambda^2$$

and  $q_1$  and  $q_2$  from Eqs. (44) and (45).

If  $\delta = \delta_*$  we have the solution in parametric form

$$a^{2} = 1 - (1 - 5F * /3)(1 - F * /3) / (1 - F *)^{4/3},$$
  

$$\delta_{*} = (F * / a^{2})(1 - F * /3) / (1 - F *)^{4/3},$$
  

$$h_{10*} = 2\delta_{*} / 3, 2q_{1}^{2} = h_{10*}^{3}, q_{2} = 0 \text{ and } \alpha_{*} = 1 / (1 - F *)^{1/3}.$$

The parameter  $F_*$  varies from 0 to 1. Fig. 5 shows the behavior of  $\delta_*(a^2)$  and  $h_{10*}(a^2)$  depending on geometry.

If  $\delta = \delta_0$  ( $\alpha = \beta = 1$ ) the solution has the simple form

$$\delta_0 = 1.5(a^2 + 1)/(3a^2 + 1), 2h_{10} = (3a^2 + 2)/(3a^2 + 1),$$
  

$$2q_1^2 = h_{10}^4, 2q_2^2 = a^8/(2a^2 + 2/3)^4.$$

These formulas are the particular case of Eqs. (C13) and (C14) with  $dh/db_{2/3} = 1/a^2$ .

Small  $a^2$  corresponds to a geometry for which the channel width stays almost constant while the depth changes significantly (sill-like channel). When  $a^2 \rightarrow 0$  we have the same solution as for a sill. Large  $a^2$  corresponds to a geometry for which the channel depth stays almost constant while the width changes significantly (contractionlike channel). When  $a^2 \rightarrow \infty$  we have the same solution as for flow through a contraction.

Fig. 6 (a–c) shows graphs of  $\xi_{10}(\delta)$ ,  $v_{10}(\delta)$ ,  $v_{20}(\delta)$ ,  $q_1(\delta)$  and  $q_2(\delta)$  for  $a^2 = 0, 1$ and  $\infty$  for the exchange regime. The range of the parameter  $\delta$  for which the exchange regime takes place for  $a^2 = 1$  is  $0 < \delta < 0.98(2.5)^{1/3}$ . With increasing  $a^2$  the lighter(denser) fluid thickness at the smallest section  $\xi_{10}(\delta, a^2)$  ( $\xi_{20}(\delta, a^2) = 1 -$ 



Fig. 7. Graphs of nondimensional discharges  $q_1$  and  $q_2$  against nondimensional upper layer thickness at the narrows  $\xi_{10}$  for channel geometry given by Eq. (42) with  $a^2 = 0$  (solid lines),  $a^2 = 1$  (dashed lines) and  $a^2 = \infty$  (heavy solid lines).

 $\xi_{10}(\delta, a^2)$ ) decreases (increases) and the lighter (denser) fluid velocity at this section  $v_{10}(\delta, a^2)$  ( $v_{20}(\delta, a^2)$ ) increases (decreases). The behavior of  $q_1(\delta, a^2)$  and  $q_2(\delta, a^2)$  is more complicated. For smaller  $\delta$  we see that  $q_1(\delta, \infty) > q_1(\delta, 1) > q_1(\delta, 0)$  and  $q_2(\delta, 0) > q_2(\delta, \infty) > q_2(\delta, 1)$ ; for larger  $\delta$  we have  $q_1(\delta, 1) > q_1(\delta, \infty) > q_1(\delta, 0)$  and  $q_2(\delta, 0) > q_2(\delta, 1) > q_2(\delta, \infty)$ . For  $a^2 > 1$  and  $\delta < 1$  the increase in  $\xi_{10}$  compensates for a decrease in  $v_{10}$  so that their product  $q_1$  stays almost the same. For  $a^2 > 1$  and  $\delta < 0.7$  the increase of  $v_{20}$  compensates for the decrease of  $\xi_{20}$  so that their product  $q_2$  stays almost the same.

Fig. 7 shows graphs of  $q_1^2$  and  $q_2^2$  vs.  $\xi_{10}$  for  $a^2 = 0, 1$  and  $\infty$ .

## 4. Discussion

The condition that the layer thicknesses continuously decrease from  $\infty$  to 0 provides a unique solution. Eqs. (31) and (33) (or in another form Eqs. (35) and (36)) are mathematical expressions of this condition. To find  $q_1(\varepsilon, \delta)$  and  $q_2(\varepsilon, \delta)$  for any given geometry h = h(b) one can solve the system Eqs. (35)–(38). Note that the system Eqs. (35)–(38) contains not only h(x) and b(x) but dh/db as well (see Eq. (35)). Therefore, for given  $\varepsilon$  and  $\delta$ , the discharge coefficients  $q_1$  and  $q_2$  depend on values of the functions h(x), b(x) and dh/db at the points  $x_1(\varepsilon, \delta)$  and  $x_2(\varepsilon, \delta)$ . We reduced the system of eight Eqs. (35)–(38) to the system of two equations Eqs. (A6) and (A7) for  $x_1$  and  $x_2$ . This system contains only the parameters  $\varepsilon$  and  $\delta$  and the functions which are fully determined by the channel geometry.

The maximum value of  $q_2(\varepsilon, \delta)$  (at  $\delta = 0$ ) is  $\varphi(x_+)$  times larger than the corresponding value of  $q_2(\varepsilon, \delta)$  (at  $\delta = 0$ ) for a sill or a contraction alone; the values of  $q_1(\varepsilon, \delta)$  at  $\delta = 0$  and at  $\delta = \delta_*$  are the same for both the contraction and the combined case but  $q_1(\varepsilon, \delta)$  is different for  $0 < \delta < \delta_*$ . The discharge  $Q_2$  reaches 0 at  $\delta = 1$  for a contraction, at  $\delta = \delta_*$  when both depth and width vary along the channel and at  $\delta = 1.5$  for a sill. If b(x) and h(x) have a minimum at the same location then  $x_+=0$  and  $\varphi(x_+)=1$ .

The solution obtained for the channel geometry Eq. (42) shows the relative effects of the changes of the channel width and depth. The parameter  $a^2$  shows how fast the depth changes compared with changes in the width,  $a^2 = db^{2/3}/dh$ . For a geometry for which the depth changes faster compared with changes in the width ( $a^2$  decreasing):

a larger relative reservoir level difference (with the same  $\varepsilon$ ) is required to arrest the denser fluid (see the graph  $\delta_*(a)$  on Fig. 5);

the velocity increases and the thickness of the denser fluid decreases at the narrowest cross-section,  $\xi_{20} = 1 - \xi_{10}$ , (Fig. 6a and b);

the value of  $\delta$ , for which  $Q_1 = Q_2 = Q_=$ , increases and the value of  $Q_=$  decreases (see Figs. 6 and 7).

In the quasi-steady approximation the volume exchange  $V = \int (Q_1 - Q_2) dt$  and the mass exchange  $M = \int (\rho_2 Q_2 - \rho_1 Q_1) dt$  do not depend on channel geometry for the flow through a contraction alone or over a sill alone. In contrast V and M are different for channels with different h(b) at  $x > x_*$  even if  $\varepsilon(t)$  and  $\delta(t)$  are the same. The theory developed in this paper allows us to predict changes of the volume exchange V

and the mass exchange M as a result of the changes of channel geometry, fluid densities and reservoirs levels.

If the relative density difference  $\varepsilon$  is small, the existence of an exchange flow and the solution depend only on the ratio of the relative reservoir level difference  $\gamma$  to the relative density difference  $\varepsilon$ ,  $\delta = \gamma/\varepsilon$ , not on the parameters  $\varepsilon = (1 - \rho_1/\rho_2)$  and  $\gamma = (1 - H_2/H_1)$  themselves. The Boussinesq (or rigid-lid) approximation can be used.

Many features of real exchange flows were neglected in this paper. We found the discharge coefficients  $q_1$  and  $q_2$  for steady flow using a simple model which does not include effects such as friction, mixing, decreasing channel width with depth, etc. To take into account effects of friction and/or decreasing channel width with depth on the discharges, correction coefficients can be introduced. Friction and mixing will change the position of the plunge point and will have a strong influence on the layer thickness where it becomes very small (see for example Brandt et al., 1996, Fig. 4), but this influence will not affect the discharge.

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# Appendix A

To solve the system Eqs. (35)–(38) it is convenient to introduce the functiins

$$\Phi(x) = (\underline{b}^{2/3})_x / \varphi_x \text{ and } \Omega(x) = \varphi \Phi + 2\underline{b}^{2/3}$$
(A1)

and denote  $\Phi_i = \Phi(x_i)$  and  $\Phi_i = \Omega(x_i)$ . Then functions  $\Phi(x)$  and  $\Omega(x)$  depend only on geometry. From Eqs. (35)–(38) we can express  $q_1, q_2, h_{1i}$  and  $h_{2i}$  in terms of  $x_i$ :

$$F_{2i} = \delta \Phi_i, \tag{A2}$$

$$F_{1i} = (1 - \delta \Phi_i) / (1 - \varepsilon \delta \Phi_i), \tag{A3}$$

$$3h_{1i} = \delta\Omega_i / (1 + \varepsilon F_{1i} F_{2i} / 2), \tag{A4}$$

$$h_{2i} = (\varphi_i - \delta \Omega_i / 3) (1 + \varepsilon F_{1i} / 2) / (1 + \varepsilon F_{1i} F_{2i} / 2)).$$
(A5)

Substituting Eqs. (A2), (A3), (A4) and (A5) into Eq. (39) we get two equations to find  $x_i(\varepsilon, \delta)$ 

$$2q_1^2/(\delta/3)^3 = \Gamma(x_1) = \Gamma(x_2),$$
(A6)

$$2q_2^2/\delta = \Pi(x_1) = \Pi(x_2), \tag{A7}$$

where

$$\Gamma(x) = (1 - \delta\Phi) \Omega^3(x) / ((1 - \varepsilon \delta\Phi)(1 + \varepsilon F_1 F_2 / 2)^3),$$
(A8)

$$\Pi(x) = \Phi\left[\varphi - (\delta/3)\Omega(x)(1 + \varepsilon F_1)/(1 + \varepsilon F_1 F_2/2)\right]^3.$$
(A9)

For any  $\delta$  in the interval (0,  $\delta_*$ ) one can calculate  $x_1$  and  $x_2$  from Eqs. (A6) and (A7). The values  $x_*$  and  $\delta_*$  can be found from

$$\delta_* \Omega(x_*) [2 + \varepsilon (1 - \delta_* \Phi_*)] = 6\varphi_*, \qquad (A10)$$

$$\Omega^{3}(x_{*})(1-\delta_{*}\Omega_{*}) = 8(1-\varepsilon\delta_{*}\Phi_{*}).$$
(A11)

It can be shown, that when  $\delta$  increases from 0 to  $\delta_*$ ,  $x_1$  decreases from  $x_+ > 0$  to  $x_* < 0$  and  $x_2$  decreases from  $\infty$  to  $x_b > 0$  (see Fig. 4).

When  $x_1 = 0$  we have from Eqs. (35)–(38) and (21)

$$\begin{split} F_{21} &= \delta, (1 - \varepsilon \delta) F_{11} = 1 - \delta, \xi_{11} = \delta (1 - \varepsilon \xi(0)), \\ \xi_{21} &= (1 - \delta) (1 - \varepsilon \xi(0)), \\ \xi(0) &= \delta (1 - \delta) / (2 - \varepsilon \delta (1 + \delta)). \end{split}$$

Then we can find corresponding values of  $\delta$  and  $x_2$  from Eqs. (A6) and (A7)

$$2q_{1}^{2}/\delta^{3} = \underline{b}^{2}(0)(1-\delta)(1-\varepsilon\xi(0))^{3}/(1-\varepsilon\delta) = \Gamma(x_{2})/27,$$
  

$$2q_{2}^{2}/\delta = \underline{b}^{2}(0)(1-\delta)^{3}(1-\varepsilon\xi(0))^{3}/(1-\varepsilon\delta)\Pi(x_{2}).$$
(A12)

In the Boussinesq approximation ( $\varepsilon = 0$ ) we have from these equations an explicit expression for  $\delta(x_2)$  and the equation to calculate  $x_2$ 

$$\delta = \left(\underline{b}^{2}(0) - \Omega^{3}(x_{2})/27\right) / (b^{2}(0) - \Phi(x_{2})\Omega^{3}(x_{2})/27$$
  
=  $\left(\left(\underline{b}^{2}(0)/\Phi(x_{2})\right)^{1/3} - \varphi(x_{2})\right) / \left(\left(\underline{b}^{2}(0)/\Phi(x_{2})\right)^{1/3} - \Omega(x_{2})/3\right)$ 

# Appendix **B**

In the Boussinesq approximation ( $\varepsilon = 0$ ) the system Eqs. (A6) and (A7) can be reduced to one equation

$$3(\Psi(x_2) - \Psi(x_1))(\Lambda^2(x_2) + \Lambda(x_2)\Lambda(x_1) + \Lambda^2(x_1)) = \Omega^3(x_2) - \Omega^3(x_1)$$
(B1)

which connects  $x_1$  and  $x_2$ . Here  $\Psi^3(x) = \Phi \varphi^3$  and  $\Lambda^3(x) = \Phi \Omega^3$ . The corresponding value of  $\delta$  is

$$\delta = \left(\Omega^{3}(x_{2}) - \Omega^{3}(x_{1})\right) / \left(\Lambda^{3}(x_{2}) - \Lambda^{3}(x_{1})\right).$$
(B2)

The value of  $x_*$  can be found from

$$\Omega^{2}(x_{*})\left(3\underline{b}^{2/3}(x_{*}) - \Omega(x_{*})\right) = 4$$
(B3)

and  $\delta_*$  is

$$\delta_* = 3\varphi_* / \Omega_* \tag{B4}$$

It follows from Eqs. (B3) and (B4) that  $1 \le \delta_* \le 1.5$ .

For a given profile h(x) and b(x) one can calculate  $x_2(x_1)$ , taking  $x_1$  from the interval  $[x_*, x_+]$  and then calculating  $x_2$  from Eq. (B1). Then one can calculate  $\delta$  from Eq. (B2) and  $q_1$  and  $q_2$  from Eqs. (A6) and (A7).

# Appendix C

Let us consider now the case when the depth and width of the channel have minima at the same location ( $x_b = 0$ ). From Eqs. (25) and (26) one can see that

$$D(0) = 0 \tag{C1}$$

Eq. (C1) together with Eqs. (37) and (38) at x = 0 give

$$F_{10}(1 - \varepsilon F_{20}) = 1 - F_{20} \tag{C2}$$

$$3h_{10}(1 + \varepsilon F_{10}F_{20}/2) = 2\delta + F_{20} \tag{C3}$$

$$h_{10}(1 + \varepsilon F_{10}/2) + h_{20} = 1 \tag{C4}$$

Now Eqs. (A6) and (A7) take the form

$$2q_1^2 = h_{10}^3(1 - F_{20}) = h_{10}^3(1 - \delta\Phi_2)$$
(C5)

$$2q_2^2 = h_{20}^2 F_{20} = h_{20}^3 \,\delta\Phi_3 \tag{C6}$$

We now have the complete system Eqs. (C2), (C3), (C4), (C5) and (C6) to find the five unknown values  $F_{10}$ ,  $F_{20}$ ,  $h_{10}$ ,  $h_{20}$  and  $x_2$ .

Using the Boussinesq approximation ( $\varepsilon = 0$ ) we can reduce the system Eqs. (C2), (C3), (C4), (C5) and (C6) to one equation for  $x_2$ 

$$3\Phi_2(\varphi_2 - \beta) / (2 + \beta^3 \Phi_2) = (\alpha^3 - 1) / (\alpha^2 + \alpha\beta + \beta^2)$$
(C7)

where the parameter  $\beta = h_{22}/h_{20}$  decreases from  $\infty$  to 0 when  $\delta$  increases from 0 to  $\delta_*$ . We also introduced

$$\alpha = h_{12}/h_{10} = \Omega_2/(2 + \beta^3 \Phi_2).$$
(C8)

The corresponding value of  $\delta$  is

$$\Phi_2 \delta = \left( \alpha^3 - 1 \right) / \left( \alpha^3 - \beta^3 \right) \tag{C9}$$

If  $\delta = 0$  we have

$$x_2 = \infty, h_{12} = 0, h_{10} = 1/3, q_1 = 0, 2q_2^2 = h_{20}^2 = (2/3)^3$$
 (C10)

If  $\delta = \delta_*$  we have

$$x_2 = x_* \le 0, h_{22} = 0, q_2 = 0, 2q_1^2 = h_{10}^3 = (2\delta_*/3)^3$$
 (C11)

The value of  $x_*$  can be found from Eq. (B3) and  $\delta_*$  from Eq. (B4).

to obtain the complete solution

$$\delta_0 = 1.5/(1+2\Phi_0), h_{10} = (1+\Phi_0/2)/(1+2\Phi_0),$$
  

$$F_{20} = 1.5\Phi_0/(1+2\Phi_0), q_1 = h_{10}^2, q_2 = F_{20}^2.$$
(C13)

Note that

$$\Phi_{0} = \begin{bmatrix} 1 & 0 \\ 0 < a_{0} < 1, & \text{when } \lim_{x \to 0} \left( \frac{dh}{db} + \frac{db}{b}^{2/3} \right) = \begin{cases} 0 & 0 \\ -1 + \frac{1}{a_{0}} & \infty \end{cases}$$
(C14)

One can see that  $h_{10} = h_{20}$  and  $q_1 = q_2$  only if  $dh/db_{2/3} = 0$  at x = 0. Varying  $\beta$  from 0 to  $\infty$  one can calculate  $x_2$  from Eq. (C7),  $\delta$  from Eq. (C9) and then  $q_1$  and  $q_2$  from Eqs. (C5) and (C6).

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