Theory of Nonlinear Motions in a Stratified Fluid
with a Free Surface

by

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ABSTRACT

The nonlinear motions in an incompressible stratified fluid with a free surface are considered. New functions are introduced which are associated with corresponding functions on the isopycnal surfaces. Simplification of the system of equations and the domain of definition is achieved with these new functions. Many new results are obtained using this system. No approximations are used in the development, but a limitation is placed on the class of allowed motions to those for which the isopycnal surfaces are single valued.

The general results are applied to obtain exact relations between integral properties of periodic gravity waves of finite amplitude and to obtain approximate equations for the large horizontal scale motion of a stratified fluid.

INTRODUCTION

description of the motion physical quantities are regarded as functions of position $\bar{x}$ and time t Alternatively, the fluid elements can be identified by their position $\bar{a}$ at some initial instant t_o and the motion specified by the subsequent position and velocity of these fluid elements This is a Lagrangian specification of the motion, the independent variables being the initial co-ordinates $\bar{a}$ and the elapsed time $t-t_o$

We introduce a new specification of the motion the independent variables being the position $\bar{x}$ and time t as in an Eulerian description, but the dependent functions associate with corresponding physical quantities on the isopycnal surfaces

Zakharov (1968) introduced the surface potential for irrotational motion of a homogeneous incompressible, inviscid fluid $\Phi^s(x,y,t)$ $\Phi(x,y,\zeta_o(x,y,t),t))$ where $z = \zeta_o(x,y,t)$ denotes the free surface, and showed that Φ^S and ζ_o are canonical variables Ostrovsky (1978) introduced the vertical displacement of isopycnal surfaces $\zeta(x,y,z,t)$ in a study of weakly nonlinear internal waves Odulo (1979) showed that an analogous function $\zeta(x,y,z,t)$ can be introduced for homogeneous fluid.

In this paper we consider the motion of a stratified incompressible fluid (§ 1.1)

In § 1.2 we introduce new functions, which are associated with corresponding functions on the isopycnal surfaces For these new functions the system of equations and the domain of definition are simpler The co·

conservation laws In § 1.4 we derive the vorticity conservation law and obtain an infinite number of conservation laws, which are generalizations of well known conservation laws in two-dimensional hydrodynamics (Arnol'd 978) Following Serrin (1959) we introduce the Clebsch representation of the velocity and give the variational principle (§ 1.5) (Benjamin & Olver 982; Henvey 1983; Luke, 1967 Salmon, 1988; Seliger & Whitham, 1968) In the particular case of "potential" motion, it is possible to give the Hamiltonian formulation of the problem (§ 1.6) (see also Voronovich 1979 Milder, 1982) In § 1.7 Hamilton's principle for the homogeneous irrotational fluid is noted (Miles 1977)

To illustrate the possibilities of the new system we consider in Chapter 2 the case of two-dimensional motions In § 2.1 we rewrite the system in simpler form and show that the problem can be reduced to one equation of this one unknown function. In § 2.2 integral relations between depth-average values are given. Simpler results are obtained in the case of the steady motion (§ 2.4) In Tables 1 and 2 integral relations are listed in detail for periodic gravity irrotational waves in homogeneous fluid In Tables 3 and 4 similar integral relations are listed for solitary irrotational waves (Starr, 1947a,b; Lonquet-Higgins, 1973, 1974 1975, 1980, 1984, 1988 Yu & Wu, 1987)

Using the new functions obtained in § 1.2, one can readily obtain approximate equations describing non-linear motions with large horizontal scale In § 3.1 the problem is introduced in dimensionless form. The system describing long gently sloping waves is noted in § 3.2 The particular cases of homogeneous fluid of variable depth and stratified

cases z-independent equations are obtained Analogous results are obtained for waves in a two-layer fluid of variable depth (§ 3.2.3 and for large scale eddies on a "β-plane" (§ 3.3)

CHAPTER 1 GENERAL THEORY

1.1 Governing equations

We will consider a layer of ideal incompressible fluid, rotating around a vertical axis z with angular velocity $\Omega/2$ We assume that in the undisturbed fluid density $\rho_o(z)$ increases with depth. The fluid is bounded above by a free surface (at $z = H + \zeta_o(x,y,t)$) and below by a rigid bottom $z = h(x,y)$ ($h_x \geq 0$, $h_y \geq 0$)

The motion of a fluid is described by

$$D^*\bar{u} + \bar{\Omega} \times \bar{u} + \rho^{-1} \bar{\nabla}_3 p + \bar{g} = 0, \tag{1.1}$$

$$\bar{\nabla}_3 \cdot \bar{u} = 0, \tag{1.2}$$

$$D^*\rho = 0 \tag{1.3}$$

with the boundary conditions

$$w = \bar{u} \quad \bar{\nabla} h \qquad \text{at } z = h(x,y)$$

$$w = D^*\zeta_o, \; p = 0 \qquad \text{at } z = H + \zeta_o(x,y,t) \tag{1.5}$$

and with corresponding initial conditions and boundary conditions respect to x and y

Here $\overline{u} = (u \quad v \quad w$ is the velocity vector ρ the density p the pressure

g the gravitational acceleratio

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$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \quad \overline{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right)$$

$$D^* \quad \frac{\partial}{} + u\frac{\partial}{} + v\frac{\partial}{} + w\frac{\partial}{} \tag{1.6}$$

1.2 New Functions and Transformed Governing Equations

We will consider only those motions, for which the equations of isopycnal surfaces are single valued functions of x and y for all t. Then we can introduce a new function $\zeta(x,y,z,t)$, representing a vertical displacement of the isopycnal surfaces (see Figure 1)

$$\rho(x,y,z + \zeta(x,y,z,t),t) = \rho_0(z) \tag{1.7}$$

In the case of a homogeneous fluid the function $\zeta(x,y,z,t)$ is defined by the equation

$$\begin{aligned} w(x,y,z + \zeta(x,y,z,t),t) &= \zeta_t + u(x,y,z + \zeta,t) \cdot \zeta_x + \\ &+ v(x,y,z + \zeta,t) \cdot \zeta_y \end{aligned} \tag{1.8}$$

We also introduce new functions (see Figure 1)

$$\begin{aligned} \mathcal{u}(x,y,z,t) &= u(x,y,z + \zeta(x,y,z,t),t) \\ \mathcal{v}(x,y,z,t) &= v(x.y.z + \zeta(x,y,z,t),t) \\ \mathcal{w}(x,y,z,t) &= w(x,y,z + \zeta(x,y,z,t),t) \\ P_n(x,y,z,t) &= p(x,y,z + \zeta(x,y,z,t),t) \\ R(x,y,z,t) &= \rho(x,y,z + \zeta(x,y,z,t),t) \end{aligned} \tag{1.9}$$

x,y,z is equal to the value of the corresponding old function at the point with the same x and y on the isopycnal surface which passed through point x,y,z in the undisturbed state

It is important to understand that independent variables stay the same But the domain of definition of the new functions is simpler For the old functions the domain of definition is (see Figure 2)

$$-\infty < x < x_o \ , -\infty < y < \infty, \ h(x,y) < z < H + \zeta_o(x,y,t) \qquad (1.10)$$

and for the new functions it is

$$\infty < x < x_o^*(y,t), \ -\infty < y < \infty, \ h^*(x,y,t) \le z < H, \qquad (1.11)$$

here h^* and x_o^* are defined by the following equations

$$h^*(x,y,t) = h(x,y) - \zeta(x,y,h^*(x,y,t),t) \quad h^*(x_o^*(y,t), \ y,t) = H \qquad (1.12)$$

If $h^* < H$ then $x_o^* = -\infty$ It is clear if $h = 0$ (flat bottom) then $h^* = 0$

and the domain of new functions is a layer of constant thickness

It is easy to obtain relations between derivatives of new and old functions

$$R_x = \rho_x + \zeta_x \rho_z \quad R_z = (1 + \zeta_z)\rho_z \text{ and so on.} \qquad (1.13)$$

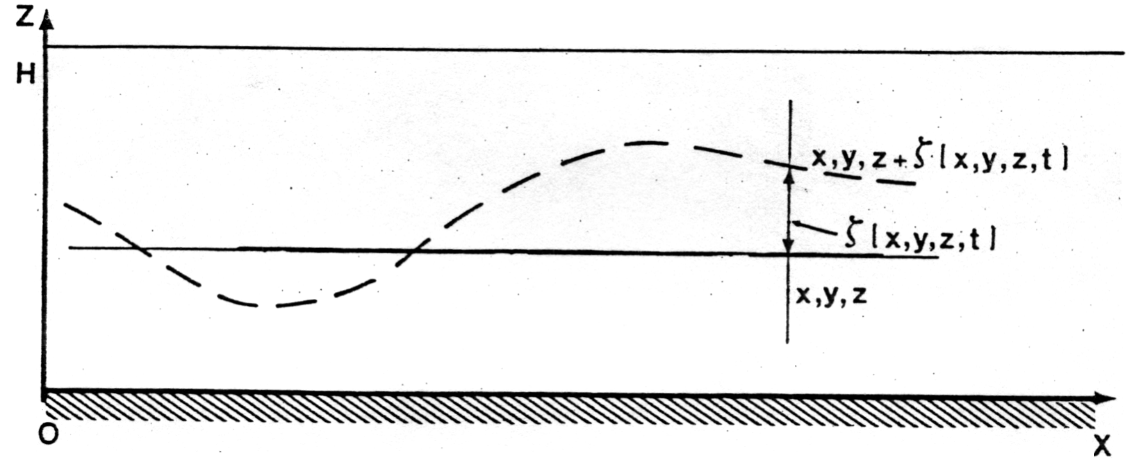

Figure 1   Undisturbed isopycnal surface (solid line) and the same isopycnal surface at time t (broken line).

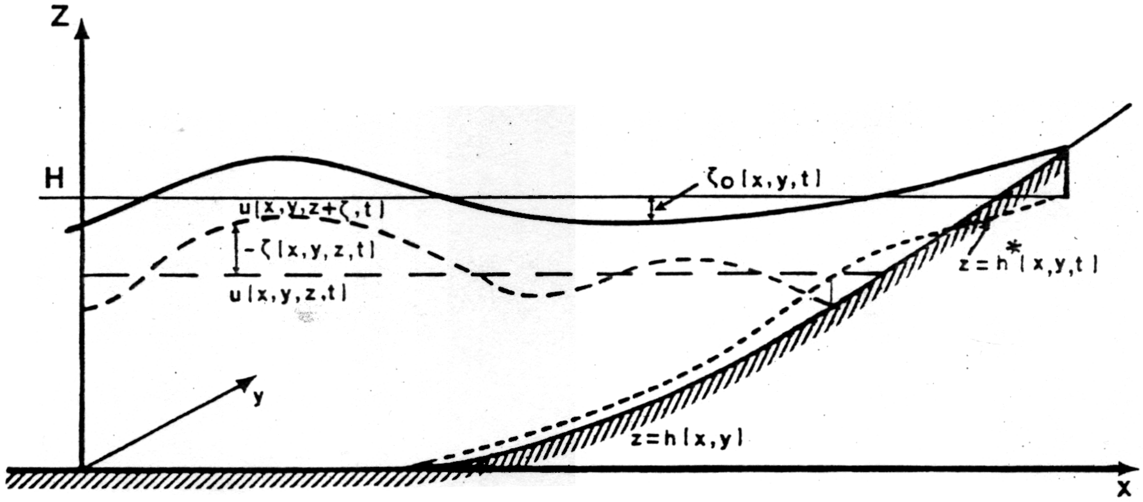

Figure 2   Schematic diagram of fluid domain.

Equations 1.1)-(1.3) are satisfied at any point in the fluid. Therefore we car change z to $z + \zeta(x,y,z,t$ in arguments functions Using (1.9), (1.13) we obtain from (1.1)-(1.5)

$$D\bar{u} + Dw \cdot \bar{\nabla}\zeta + \frac{1}{\rho_o} \bar{\nabla} P + \bar{\Omega} \times \bar{u} = 0, \tag{1.14}$$

$$(1 + \zeta_z) Dw + \frac{1}{\rho_o} P_z + N^2\zeta = 0, \tag{1.15}$$

$$\zeta_{zt} + \bar{\nabla} \cdot [\bar{u} (1 + \zeta_z) = 0, \tag{1.16}$$

$$w = D\zeta \tag{1.17}$$

From (1.7), (1.9) it follows, that

$$R (x,y,z,t) = \rho_o(z) \tag{1.18}$$

Boundary conditions are

$$\zeta_t = \bar{u} \cdot \bar{\nabla} (h(x,y) - \zeta) \quad \text{at } z = h*(x,y,t) \tag{1.19}$$

$$P = g\rho_o\zeta, \ \zeta = \zeta_o \qquad \text{at } z = H. \tag{1.20}$$

Here $\bar{u} = \{u,v,0\}$, $N^2 = -g\rho_o' / \rho_o$, $P = P_n + g\rho_o\zeta - g\int_z^H \rho_o dz$,

$$D = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \tag{1.21}$$

If we consider a homogeneous fluid, system (1.14) (1.17) changes a little equations (1.16), (1.17) stay the same, in (1.14) - (1.15) we take $\rho_o(z)$ = const ($N^2 = 0$). Clearly (1.8) and (1.17) are the same. The problem (1.14) (1.20) is simpler than the problem (1.1)-(1.5) in the following ways:

The domain of definition of the new functions is simpler In the

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system in a simple fixed region

2 Operator D 1.21 is simpler then D^* (1.6) and equation (1

contains no differentiation with respect to z

3 Function w is simply related to u v and by ζ(1.17)

4 Instead of a problem with six (u,v,w,ρ,p,ζ_o) unknown functions we have obtained a problem with four (u,v,P,ζ) unknown functions

1.3 The Impulse and Energy Conservations Laws

Let $\eta = z + \zeta$ (1.22)

Henceforth in this chapter we will consider only the case h = 0 Using (1.15 to eliminate Dw from 1.14) we get

$$\eta_z Du + \frac{1}{\rho_o} (\eta \; P_x - \eta_x P_z \;\; z \;\; \eta_x N^2 - \Omega v \eta_z = 0$$

$$\eta_z D \quad \frac{1}{\rho_o} (\eta_z P_y - \eta_y P_z) - \zeta \eta_y N^2 - \Omega u \eta_z = 0 \tag{1.24}$$

Multiplying equations (1.23),(1.24) and (1.15) by nu^{n-1} nv^{n-1} and nw^{n-1} respectively, and then adding to each (1.16) multiplying by u^n v^n and w^n respectively we obtain

$$a_{n_t} + a_{n+1_x} + (v\ a_n)_y + n\,\frac{a_{n-1}}{\eta_z} \left[\frac{1}{\rho_o} (P_x\eta_z - P_z\eta_x) - N^2\zeta\eta_x\right] - \Omega v n a_{n-1} = 0, \tag{1.25}$$

$$b_{n_t} + b_{n+1_x} + (u\ b_n)_y + n\,\frac{b_{n-1}}{\eta_z} \left[\frac{1}{\rho_o} (P_y\eta_z - P_z\eta_y) - N^2\zeta\eta_y\right] + n\Omega u b_{n-1} = 0, \tag{1.26}$$

$$c_{n_t} + (uc_n)_x + (v\ c_n)_y + n\,\frac{c_{n-1}}{\eta_z} \left[\frac{1}{\rho_o} P_z + N^2\zeta\right] = 0, \tag{1.27}$$

where $a_n = \rho_o\ \eta_z\ u^n, b_n = \rho_o\ \eta_z\ v^n,\ c_n = \rho_o\ \eta_z\ w^n$

Integrating (1.25) (1.26 over z from 0 to H for n = 1 we obtain

$$I^x_t + S^{xx}_x + S^x_y \quad \Omega \int_o b_1 dz = 0 \tag{1.28}$$

$$I^y + S^{yx}_x + S^{yy}_y \quad \Omega \int_o^H a_1 dz = 0 \tag{1.29}$$

In the case $\Omega = 0$ these equations are impulse conservation laws.

Here

$$I^x = \int_o^H \rho_o \eta_z u dz , \qquad I^y = \int_o^H \rho_o \eta_z v dz$$

$$S^{xx} = \int_o^H [\eta_z(\rho_o u^2 + P) - \frac{1}{2} \rho_o N^2 \zeta^2] dz - \frac{1}{2} g\rho_o(H)\zeta_o^2 ,$$

$$= \int_o^H [\eta_z(\rho_o v^2 + P) - \frac{1}{2} \rho_o N^2 \zeta^2] dz - \frac{1}{2} g\rho_o(H)\zeta_o^2$$

$$S^{yx} = \int_o^H \eta_z \rho_o uv dz$$

Adding (1.25)-(1.27) for n = 2, we obtain

$$E_t + \bar{\nabla} \cdot \quad \bar{u}\eta_z(T + P) + (P\eta_t)_z = 0 \tag{1.30}$$

Where

$$E = \eta_z T + \pi^i , \quad 2T = \rho_o(u^2+v^2+w^2), \quad \pi^i = 1/2 \ \rho_o N^2 \zeta^2$$

Integrating (1.30) over z from 0 to H, and using boundary conditions

$$E_t + F^x_{\ x} + F^y_{\ y} = 0 \tag{1.31}$$

Here $E = K + \Pi^i + \Pi^s$ $\quad K = \int_0^H \eta_z T dz$

$$\Pi^i = \int_0^H \pi^i dz, \quad \Pi^s = \frac{1}{2} g\, \rho_o(H)\zeta_o^2$$

$$F^x = \int_0^H u\eta_z(T+P)dz, \quad F^y = \int_0^H v\eta_z(T+P)dz$$

It is important to note that the old functions do not allow separation of kinetic energy K and the baroclinic potential energy Π^i But the functions make this separation easy

1.4 The Vorticity Conservation Law

Now we will obtain an infinite number of conservation laws, which are generalizations of well known conservation laws in two dimensional hydrodynamics (Arnol'd, 1978).

We introduce the new functions

$$\bar{V} = \ U, V, W\} = \rho_o\ (\bar{u} + w\ \nabla_3\bar{\eta}) \tag{1.32}$$

From (1.14) - (1.15) we find after manipulation

$\bar{\eta}$ $\bar{\omega}$ o $\zeta)\bar{k}$

Here $\bar{\omega} = (\omega_1 \ \omega_2 \ \omega_3 = \bar{\nabla}_3 \times \bar{V}$ - vorticity $\Lambda = \bar{u} \cdot \bar{V}$ T P; $\bar{k} = (0,0,1)$

By taking the curl of (1.33) we can show that

$$\bar{\omega}_t + \bar{\nabla}_3 \times ((\bar{\omega} + \rho_o \bar{\Omega}) \times \bar{u} + \frac{\rho_o}{\rho_o} T + \rho_o N^2 \zeta)\bar{k} = 0 \qquad (1.34$$

The projection of (1.34 on the axis z has the form

$$\omega_{3t} + [u(\omega_3 + \rho_o \Omega)_x + v(\omega_3 + \rho_o \Omega)]_y = 0 \qquad (1.35$$

This equation is a law of conservation of the vertical component of vorticity

Using (1.16) we can obtain an infinite number of conservation laws

$$\eta_z \Phi(\frac{\omega_3 + \rho_o \Omega}{\eta_z} _t + \bar{\nabla} \cdot \ \bar{u} \eta_z \Phi(\frac{\omega_3 + \rho_o \Omega}{\eta_z} = 0 \qquad (1.36)$$

Here Φ is an arbitrary function. We obtain conservation laws (1.34) (1.36) at the expense of a limitation of the class of allowed motions The initial system (1.1)-(1.3) does not have these conservation laws

1.5 Variational principal Clebsch representation

Henceforth, except Chapter 3, we will consider the case $\Omega=0$. It is then possible to introduce the Clebsch representation (Seliger and Whitham

$$\overline{V} = \overline{\nabla}_3\phi + \lambda\overline{\nabla}_3\mu + \chi\overline{\nabla}_3\rho_o \tag{1.37}$$

where ϕ, λ, μ and χ are new unknown functions

Then the vector vorticity is

$$\overline{\nabla}_3\lambda\times\overline{\nabla}_3\mu + \overline{\nabla}_3\chi\times\overline{\nabla}_3\rho_o \tag{1.38}$$

From (1.37) and (1.33) we have

$$\overline{\nabla}\Lambda + D\lambda \cdot \overline{\nabla}\mu + \lambda \cdot \overline{\nabla}D\mu + [D\chi + T/\rho_o - g\zeta]\overline{\nabla}\rho_o = 0 \tag{1.39}$$

where

$$\Lambda = D\phi - T + P. \tag{1.40}$$

By (1.37) we have introduced four functions (ϕ, λ, μ, χ) in place of three

(u, v, w)

Therefore we can put

$$D\chi + T/\rho_o - g\zeta = 0 \tag{1.41}$$

Then from (1.38) we can find

$$\Lambda + \lambda D\mu = R(\mu,\lambda,t) \tag{1.42}$$

and

$$R_\lambda = D\mu, \tag{1.43}$$

$$R_\mu = -D\lambda. \tag{1.44}$$

Where R is an arbitrary function for which dR

dt ∂t

In (1.14 - (1.16) replacing $\bar{u}$ and w by ϕ λ, μ, χ and η we obtain

together with (1.40) (1.42) a system of five equations in five unknown functions This system follows from the variational principle

$$\delta \iiint\limits_{-\infty}^{\infty} \int\limits_{0}^{H} P(\phi,\lambda,\mu,\chi,\eta)\eta_z dzdxdydt = 0 \tag{1.46}$$

Here

$$P = R + T - D\phi - \lambda D\mu \tag{1.47}$$

In 1.46) λ, μ, η, χ, ϕ are varied independently to give

$$D\mu = R_\lambda \tag{1.43}$$

$$D\lambda = - R_\mu \tag{1.44}$$

$$D\chi + T/\rho_o - g\zeta = 0, \tag{1.41}$$

$$w = D\eta \tag{1.17}$$

$$\eta_{zt} + \bar{\nabla}(u\eta_z = 0 \tag{1.16}$$

1.6 "Potential" motions. Hamiltonian description

We can consider motions for which

$$R = \lambda = \mu = 0 \tag{1.48}$$

We will call such motions "potential" In this case the problem is reduced to the system

$$\eta_t = w - \bar{u} \cdot \bar{\nabla}\eta \tag{1.50}$$

where the dependence of ϕ on χ and η is defined by the problem

$$(w - \bar{u}\cdot\bar{\nabla}\eta)_z + (u\eta_z)_x + (v\eta_z)_y = 0 \tag{1.51}$$

with boundary conditions

$$\eta = 0 \qquad\qquad \text{at } z = 0, \tag{1.52}$$

$$\rho_o g\zeta - T = 0 \qquad \text{at } z = H \tag{1.53}$$

We can rewrite the system (1.49) (1.50) in a canonical form (Voronovich, 1979; Milder, 1982

$$\chi_t \quad \frac{\delta H}{\delta \eta} \quad \eta_t \quad \frac{\delta H}{\delta \chi} \tag{1}$$

Here the Hamiltonian is

$$H = \iint^{\infty} \int_o^H (\eta_z T + \frac{1}{2}\rho_o N^2 \zeta^2)dz + \frac{1}{2} g\rho_o(H)\ \zeta_o{}^2\ ]dxdy \tag{1.55}$$

Note, that (1.51) can be written in the form

$$\frac{\delta H}{\delta \phi}$$

7 Irrotational motions of homogeneous fluid

Now we add to the conditions (1.48)

$$\rho_o = 1, \ \chi = 0 \tag{1.56}$$

corresponding to an irrotational motion of homogeneous fluid. In this case

rewritten in the form

$$L(\phi,\eta) \equiv \frac{\phi_z}{\eta_z}(1+\zeta_x^2+\zeta_y^2)-\phi_x\eta_x-\phi_y\eta_y \ +(\phi_x\eta_z-\phi_z\eta_x)_x+(\phi_y\eta_z-\phi_z\eta_y)_y = 0 \tag{1.57}$$

$$\eta_t+\phi_x\eta_x+\phi_y\eta_y-\frac{\phi_z}{\eta_z}(1+\zeta_x^2+\zeta_y^2) = 0 \tag{1.58}$$

boundary conditions

$$\eta = 0 \qquad \text{at } z = 0 \tag{1.59}$$

$$\phi_t + \frac{1}{2}(\phi_x^2+\phi_y^2) - \frac{1}{2}\frac{\phi_z^2}{\eta_z^2}(1+\zeta_x^2+\zeta_y^2 + g\zeta = 0 \text{ at } z = H \tag{1.60}$$

Integrating by parts

$$\iint_{-\infty}^{\infty}\int_0^H \phi \cdot L(\phi \quad \eta)\, dzdxdy = 0 \tag{1.61}$$

we obtain

$$2T = 2\iint_{-\infty}^{\infty}\int_0^H \eta_z T dzdxdy = \iint_{-\infty}^{\infty}(\phi[\frac{\phi_z}{\eta_z}(1+\zeta_x^2+\zeta_y^2)-\phi_x\zeta_x-\phi_y\zeta_y \quad _{z=H} dxdy \tag{1.62}$$

Using (1.58) we have

$$2T = \iint_{-\infty}^{\infty} \phi\eta_t \quad _{z=H} dxdy \tag{1.63}$$

means that the kinetic energy of the entire fluid volume can be expressed in terms of functions and their derivatives at the free surface. Also the potential energy can be expressed through ζ_o as

$$2\overline{\Pi}^{s} = \quad \int\int^{\infty} \qquad 1.64$$

Hence the Hamiltonian can be expressed by means of the functions and their derivatives at the free surface

$$H = T + \overline{\Pi}^{s} \qquad (1.65)$$

The equation (1.60) and the equation (1.58) at z = H can be written in canonical form (Miles (1977))

$$\phi_t = -\frac{\delta H}{\delta \eta} \quad \eta_t = \frac{\delta H}{\delta \phi} \qquad (1.66)$$

Chapter 2. TWO-DIMENSIONAL MOTIONS

2.1 Governing Equations and Reducing the System to one Equation.

In this chapter we will consider two-dimensional motion in a fluid of constant depth without rotation. In this case the problem (1.14) - 1.20) has the form

$$u_t + uu_x + \eta_x (w_t + uw_x) + \frac{1}{\rho_o} P_x = 0, \tag{2.1}$$

$$+ u\, w_x) + \frac{1}{\rho_o} P_z + N^2\zeta = 0, \tag{2.2}$$

$$+ (u\eta_z)_x = 0, \tag{2.3}$$

$$w = \eta_t + u\eta_x, \tag{2.4}$$

with boundary conditions

$$P = g\rho_o\zeta_o \quad \text{at } z = H \tag{2.5}$$

$$\eta = 0 \quad \text{at } z = 0 \tag{2.6}$$

The corresponding problem for functions $\overline{V}$, η P has the form

$$U_t + \frac{1}{2\rho_o}(U^2 - W^2 \frac{1 + \eta_x^2}{\eta_z^2}) + P]_x = 0 \tag{2.7}$$

$$W_t + \frac{1}{\rho_o} [UW - W^2 \frac{\eta_x}{\eta_z}]_x + P_z + \rho_o N^2 \zeta = 0, \tag{2.8}$$

$$+ [U\eta_z - W\eta_x]_x = 0 \tag{2.9}$$

$$W (1 + \eta_x^2) = \eta_z(\rho_o\eta_t + U\eta_x). \tag{2.10}$$

the ame boundar conditio

From 2.7 - (2.8) we can obtain

$$\omega_t + u\omega - \rho_o(- g\zeta + \frac{u^2 + w^2}{2})]_x = 0$$

where vorticity ω is given by

$$\omega = U_z - W_x = (\rho_o u)_z + (\rho_o \eta_x w)_z - \rho_o (\eta_z w)_x \tag{2.12}$$

Equation (2.11) is the y-component of (1.34) for the plane case without rotation

We will say that an equation has the form of a conservation law, if it has the form

$$F_t + \Phi_x = 0 \tag{2.13}$$

It is clear that we can introduce a new function ϕ

$$F = \phi_x \tag{2.14}$$

to obtain the expression

$$\phi_t + \Phi = C(z,t). \tag{2.15}$$

This means that value of $\phi_t + \Phi$ at time t is uniform on an isopycnal surface

Equations (2.3) or (2.9), (2.7) and (2.11) have the form of (2.13) We introduce the functions θ and ϕ

$$\eta - z = \zeta = \theta_x, \tag{2.16}$$

$$U = \phi_x, \tag{2.17}$$

to obtain from (2.3), (2.7) and (2.11) the following system

$$\theta_{tz} + u\ (\theta_{xz} + 1) = Q(z,t), \tag{2.18}$$

$$+ \eta.$$

$$\phi_{zt} - W_t \quad u\omega - \rho_o \left(- g\zeta + \frac{u^2 + w^2}{2}\right. \quad = \Omega(z$$

where functions Q and E can be determined from boundary conditions at x, and

$$\Omega(z,t) = E_z(z,t) \tag{2.21}$$

It is possible to reduce the system (2.1 (2.4 to an equation for the function θ Indeed we can express functions u, w and η in terms θ

$$\eta = \theta_x + z \tag{2.16}$$

$$u = \frac{Q - \theta_{tz}}{1 + \theta_{xz}} \tag{2.18}$$

$$w = \theta_{xt} + \frac{\theta_{tz}}{1 + \theta_{xz}} \theta_{xx} \tag{2.4'}$$

Substituting (2.12) (2.16), (2.18) and (2.4') into equation (2.11) we obtain the equation for θ (x,z,t)

2.2 Integral Relationships

The integral properties are of particular interest We will define depth-averaged functions

$$> \equiv \frac{1}{H} \int_o^H . \quad dz \tag{2.22}$$

From system (2.1) - (2.4) we can obtain the following equations $(n=0,1,2,\ldots)$

$(\eta^n \eta_z)_t \qquad \eta^n\eta_z \qquad \eta^n$

$$(\rho_o u\eta^n \eta_z)_t + (\rho_o u^2 \eta^n \eta_z)_x + (P \eta^n \eta_z)_x - (P \eta^n \eta_x)_z$$

$$- \rho_o N^2 \zeta\eta^n \eta_x = \rho_o uw (\eta^n)_z \tag{2.24}$$

$$(\rho_o w \eta^n \eta_z)_t + (\rho_o uw \eta^n \eta_z)_x + \eta^n (P_z + \rho_o N^2 \zeta) = w^2 \eta^{n-1} \eta_z \tag{2.25}$$

Integrating these equations with respect to z from 0 to H, we obtain the following equations (n = 1,2,3)

$$<\rho_o \eta_z>_t + <\rho_o u \eta_z>_x = 0 \tag{2.26}$$

$$\eta_{o_t} + <u\eta_z>_x = 0 \tag{2.26'}$$

$$<\rho_o \eta^n \eta_z>_t + <\rho_o u \eta^n \eta_z>_x = n <\rho_o w \eta^n \eta_z> \tag{2.27}$$

$$<\eta^n \eta_z>_t + <u\eta^n \eta_z>_x = n <w \eta^n \eta_z> \tag{2.27'}$$

$$<\rho_o u\eta_z>_t + [<\rho_o u^2\eta_z> + <P\eta_z> - g <\rho_o\zeta\zeta_z>]_x = 0 \tag{2.28}$$

$$<\rho_o u\eta^n\eta_z>_t + [<\rho_o u^2\eta^n\eta_z> + <P\eta^n\eta_z> + g(<\rho_o\zeta\eta^n\eta_z>$$

$$-<\rho_o\eta^{n+1}>/(n+1))]_x = n<\rho_o uw_\eta{}^{n-1}\eta_z> \tag{2.29}$$

$$H< \rho_o w\eta_z>_t + H<\rho_o uw\eta_z>_x = P_o - g\zeta_o \tag{2.30}$$

$$<\rho_o w\eta^n\eta_z>_t + <\rho_o uw\eta^n\eta_z>_x = \tag{2.31}$$

$$= n[<\rho_o w^2\eta^{n-1}\eta_z> + <P\eta^{n-1}\eta_z> - g<\rho_o\zeta\eta^{n-1}\eta_z>] - g <\rho_o\eta^n\zeta_z>.$$

If we put $\dfrac{\partial}{\partial y} = 0$ and $\Omega = 0$ in Equation (1.28), we obtain Equation (2.28) The energy conservation law (1.30) and (1.31) in this case $(\dfrac{\partial}{\partial y} = 0)$ can be rewritten in the forms

$$\frac{1}{2} [\rho_o\eta_z(u^2+w^2)-g\rho_o{}'\zeta^2]_t + (\rho_o\frac{u^2 + w^2}{2} + P)u\eta_z]_x+(P\eta_t)_z = 0 \tag{2.32}$$

Values of functions on the free surface we will denote as

$$f\ (x,H,t) = f_s\ \ x,t) \tag{2.34}$$

and on the bottom as

$$f\ (x,0,t) = f_b\ (x,t) \tag{2.35}$$

From (2.1) - (2.10) we can obtain

$$U_{s_t} + (U_s \quad \frac{u_s^{\ 2} + w_s^{\ 2}}{2} + g\ \rho_{os}\ \zeta_o)_x = 0 \tag{2.36}$$

$$w_s = \zeta_{o_t} + u_s\ \zeta_{o_x} \tag{2.37}$$

$$\rho_{o_b}\ (u_{b_t} + u_b\ u_{bx}) + P_{bx} = 0 \tag{2.38}$$

$$P_{zb} = 0 \qquad 2.39$$

It is useful to introduce a function

$$I(x,z,t) = \int_o^z u\eta_z dz \tag{2.40}$$

From (2.3) we have

$$\eta_t + I_x = 0 \tag{2.41}$$

and using (2.4) we obtain

$$w = u\ \eta_x - I_x. \tag{2.42}$$

From (2.1) - (2.6), (2.11 and (2.12) we can obtain

$$H \langle\omega\rangle = U_s - \rho_{o_b} u_b - H \langle\rho_o w \eta_z\rangle_x, \qquad (2.43)$$

$$H \langle\omega\eta\rangle = U_s \eta_s - H\langle\rho_o w\eta \eta_z\rangle_x - n\langle\rho_o u\eta^{n-1} \eta_z\rangle, \qquad 2.44$$

$$H \langle u\omega\rangle = u_s U_s - \frac{1}{2} \rho_{o_b} u_b^2 - H\langle\rho_o uw\eta_z\rangle_x - \rho_{os}\frac{u_s^2+w_s^2}{2} + \frac{H}{2} \langle\rho_o' (u^2+w^2)\rangle, \qquad (2.45)$$

$$\langle u\eta_z\rangle U_s - \langle\rho_o(u^2 + w^2)\eta_z\rangle - \langle\rho_o I w\eta_z\rangle_x \qquad (2.46)$$

We can also obtain the equation

$$)_t + (u\omega\eta^n)_x = \eta^n \rho_o \left[\frac{u^2 + w^2}{2} - g\zeta\right] \quad {}^{n-1}\omega \qquad (2.47)$$

which gives the following integral relationships

$$\langle\omega\eta^n\rangle_t + \langle u\omega\eta^n\rangle_x = \langle\frac{u^2+w^2}{2} \eta^n \rho_o - g \langle\zeta\eta^n \rho_o + \quad {}^{n-1}$$

2.3 "Potential" motions

Consider motions for which velocity can be represented in the form

$$V = \phi_x, \; W = \phi_z + \chi\rho_o \qquad (2.49)$$

From (2.12) we have

$$\omega = - \chi_x\rho_o \qquad (2.50)$$

Hence from (2.11) we have

$$\chi_t + u\chi_x + \frac{u^2 + w^2}{2} - g\zeta = 0 \qquad (2.51)$$

This equation together with (2.9) and (2.10) gives us three equations

ıree funct ϕ d η see (

2.4 STEADY MOTIONS

2.4.1 Governing Equations

All equations become much simpler if we consider steady motions, is, motions for which

$$f(x,z,t) = f(x-ct,z) \text{ or equivalently } f_t = -cf_x \tag{2.52}$$

and we can integrate all equations which have the form (2.13) Indeed, we have from (2.1) (2.4)

$$\rho_o \frac{(u-c)^2 + w^2}{2} + P = E(z) \tag{2.53}$$

$$Q(z)w_x + \frac{1}{o} P_z + N^2\zeta = 0 \tag{2.54}$$

$$\eta_z(u-c) = Q(z) \tag{2.55}$$

$$\eta_z w = Q(z)\ \eta_x \tag{2.56}$$

Equation (2.30) is the Bernoulli equation.

From (2.11) we have

$$(u-c)\omega - \rho_o N^2\zeta - \rho_o \frac{u^2 + w^2}{2} = \Omega(z) \tag{2.57}$$

Where (see (2.21)

$$\Omega(z) = E'(z) \tag{2.58}$$

All variables in (2.57) may be readily expressed in terms of η

$$\frac{Q}{\eta_z} \qquad 2.59$$

$$w = Q\frac{\eta_x}{\eta_z} \qquad (2.60)$$

$$P = E - Q^2\rho_o\frac{1+\eta_x^2}{\eta_z^2} \qquad (2.61)$$

$$\omega = \left[\rho_o Q\frac{1+\eta_x^2}{\eta_z}\right] - \rho_o Q\,\eta_{xx} \qquad (2.62)$$

Now from (2.57) (or from (2.54)) we obtain an equation for η:

$$Q^2\rho_o\left[\frac{\eta_x}{\eta_z}\right]_x \quad \frac{1}{2}\left[\rho_o\frac{1+\eta_x^2}{\eta_z^2}\right]_z + \rho_o N^2\zeta + E'(z) = 0 \qquad (2.63)$$

We must solve this equation with the boundary conditions

$$\frac{\rho_o Q^2}{} \quad \frac{1+\eta_x^2}{\eta_z^2} + g\rho_o\zeta = E \text{ at } z = H, \qquad (2.64)$$

$$\eta = 0 \qquad \text{at } z = 0, \qquad (2.65)$$

and some boundary conditions in x.

2.4.2 Integral relationships

Rewrite Equations (2.22) - (2.24) for steady motion

$$c)\ \eta \quad _z]_x = w\ (\eta^n)_z \qquad (2.6!$$

$$(u - c)\ \eta_z\ \eta^n + \eta^n\ P\ \eta_z]_x - \rho_o\ N^2 \zeta\ \eta^n\ \eta_x - (\eta^n\ P\ \eta_{x})_z$$

(2.66)

$$n\ \rho_o\ uw\eta^{n-1}\ \eta_z$$

$$\rho_o\ (u - c)\ w\ \eta_z\ \eta^n{}_x + P_z\ \eta^n + \rho_o\ N^2\ \zeta\ \eta^n = \rho_o n\ \eta^{n-1}\ \eta_z\ w^2 \qquad (2.67)$$

Integrating over z from 0 to H we have (n = 1,2,

$$\langle u\ n_z\rangle = c\ \frac{\eta_o}{H} + Q_o \quad Q_o = \langle Q(z)\rangle \qquad (2.68)$$

$$\langle u\ \eta^n\ \eta_z\rangle_x \quad \frac{}{H}\ \eta_o^n\ \zeta_{o_x} + n\ \langle w\ \eta^{n-1}\ \eta_z\rangle \qquad (2.69)$$

$$\langle \rho_o\ (u - c)\ \eta_z\rangle = Q_1, \quad Q_1 = \langle \rho_o\ Q(z)\rangle \qquad (2.70)$$

$$(u - c\quad \eta^n\ \eta_z\rangle_x = n\ \langle \rho_o\ w\ \eta^{n-1}\ \eta_z\rangle \qquad (2.71)$$

$$\langle \rho_o\ u^2\ \eta_z\rangle - c\ \langle \rho_o\ u\ \eta_z\rangle + \langle P\ \eta_z\rangle - g\ \langle \rho_o\ \zeta\ \zeta_z\rangle = R_o \qquad (2.72)$$

$$u^2\ \eta^n \eta_z\rangle - c\ \langle \rho_o\ u\ \eta^n \eta_z\rangle + \langle P\ \eta^n\ \eta_z\rangle$$

(2.73)

$$+ g\ \langle \rho_o \left[\zeta\ \eta^n\ \eta_z \quad \frac{1}{n+1}\ \eta^{n+1} \right\rangle = n\ \langle \rho_o\ uw\eta^{n-1}\ \eta_z\rangle$$

$$H\ Q\ (z)\ \langle w\rangle_x = P_b - g\ H\ \langle \rho_o \zeta_z\rangle \qquad (2.74)$$

$$\langle \rho_o\ (u-c\ \ w\eta^n \eta_z\rangle_x = n\langle (P+\rho_o w^2)\eta^{n-1}\ \eta_z\rangle - g\langle \rho_o(\eta^n \eta_z + n\zeta\eta^{n-1}\eta_z - \eta^n)\rangle. \qquad (2.75)$$

The energy conservation law (2.33) has the form

$$\frac{u^2+w^2}{} \eta \quad + g \quad \rho_o \quad > = \frac{1}{c} \quad u\eta_z(\rho_o \frac{u^2+w^2}{}$$

From equations (2.35) - (2.40) we have

$$U_s\ (u_s-c) - \rho_{os} \frac{u_s^2+w_s^2}{2} + g\rho_{os}\zeta_o = E_o \quad \frac{c}{2} \tag{2.77}$$

$$w_s = (u_s - c)\ \zeta_{o_x} \tag{2.78}$$

$$\rho_{ob}\ u_b\ (u_b - c) + P_b = E_o \quad \frac{c^2}{2} \tag{2.79}$$

$$I = c\ \eta + q\ (z),\ q\ (z) = \int_o^z Q(z)\ dz \tag{2.80}$$

Using equations (2.55) - (2.57) we can obtain the following expressions

(n = 0,1,2

$$<\rho_o\ u\eta^n\ \eta_z> = <\rho_o Q\eta^n> + c\ <\rho_o\eta^n\ \eta_z> \tag{2.81}$$

$$<\rho_o\ w\eta^n\ \eta_z> = \frac{1}{n+1}\ <\rho_o Q\eta^{n+1}>_x \tag{2.82}$$

$$<P\eta^n\ \eta_z> = <(E_o - \frac{c^2}{2}\ \rho_o)\eta^n\ \eta_z> - <\rho_o \frac{u^2+w^2}{2}\ \eta^n\ \eta_z> + c<\rho_o u\eta^n\ \eta_z> \tag{2.83}$$

$$<\rho_o\ u^2\ \eta^n\ \eta_z> = c\ <\rho_o u\eta^n\ \eta_z> + <\rho_o Qu\eta^n> \tag{2.84}$$

$$<\rho_o\ uw\eta^n\eta_z> = c\ <\rho_o w\eta^n\eta_z> + <\rho_o\ Qw\eta^n> \tag{2.85}$$

$$<\rho_o w^2\ \eta^{n-1}\ \eta_z> = \frac{1}{n}\ <\rho_o Qw\eta^n>_x - n\ <P\eta^{n-1}\ \eta_z> + \tag{2.86}$$

$$g(n \quad {}_{o}\zeta \eta^{n} \eta_{z} \qquad \eta^{n} \eta_{z} \qquad {}^{n}$$

$$\langle u\, P\, \eta^{n}\, \eta_z \rangle = c \langle P\, \eta^{n}\, \eta_z \rangle + \langle QP\, \eta^{n} \rangle \tag{2.87}$$

2.4.3 Steady Motion of a Homogeneous Fluid

Considering a motion of a homogeneous fluid we can put $\rho_o = 1$ From Equation (2.57) we have

$$\omega\ (u-c) = \Omega(z), \tag{2.88}$$

and taking into account (2.51) we obtain

$$\omega = \frac{E\ (z)}{Q(z)}\ \eta_z \tag{2.89}$$

The problem can be reduced to the following equation (see Equation (2.58)):

$$\left(Q\ \frac{1 + \eta_x^{2}}{\eta_z}\right)_z - Q\eta_{xx} = \frac{E}{Q}\ \eta_z, \tag{2.90}$$

with boundary conditions (2.60), (2.61)

Henceforth we will consider the easiest case: irrotational flow ($\omega=0$) and assume that

$$E(z) = E_o, \tag{2.91}$$

$$Q(z) = -\ c. \tag{2.92}$$

Equations (2.53) - (2.56) then take the form

$$\frac{(\iota - c)^2 + w^2}{}$$

$$P_z = cw_x, \tag{2.94}$$

$$(u - c)\, \eta_z = - c, \tag{2.95}$$

$$\eta_z\, w = - c\, \eta_x, \tag{2.96}$$

with boundary conditions

$$\eta = 0 \qquad \text{at } z = 0, \tag{2.97}$$

$$P = g\, \zeta_o \qquad \text{at } z = H. \tag{2.98}$$

From (2.93) - (2.96) we obtain an equation for η

$$\left[\frac{1 + \eta_x^2}{\eta_z}\right]_z = \eta_{xx} \qquad 99$$

with boundary conditions

$$\eta = 0 \qquad \text{at } z = 0,\ H \tag{2.100}$$

$$c^2\, \frac{1 + \eta_x^2}{2\, \eta_z^2} + g\eta = E_o + gH \quad \text{at } z = H \tag{2.101}$$

Using the assumption that the motion is irrotational we obtain

$$U_z = - c\, \eta_{xx}, \tag{2.102}$$

$$(\zeta U)_z = c\zeta_z - U - \frac{1}{2}\, c\zeta^2_{\ xx}, \tag{2.103}$$

$$\mathrm{\beta U} \quad \frac{1}{} \quad u^2 + w^2 \quad - c \quad u\eta_x \qquad 2.104$$

$$(\eta\, U)_z = c\; n\; \eta \quad \zeta_z \quad \frac{c}{n+1} (\eta^{n+1})_{xx} \qquad (2.105)$$

Equations (2.93 (2.96) and boundary conditions (2.97) (2.98) contain our problem From there we can obtain expressions for other quantities (see Table 1 first column) In the second column of Table 1 there are depth-average quantities We see that all depth-average quantities are expressed through ζ_o, ζ_{ox} and $<\zeta>_{xx}$ The value c can also be calculated by $_o$ and ζ_{ox} (see below)

Consider periodic waves (see Figure 3) Introduce an average over half the wave length

$$f(x,z) = \frac{2}{L} \int_{-L/2}^{o} f\ (x,z\ \ dx \qquad 2.106$$

third column of Table 1 contains these period-average values The asterisk marks values which are obtained independent of z This means that are the same on all isopycnal surfaces

In Column A of Table 1 there are relationships between functions, which are dependent on x and z. In column B there are functions independent of z. On the top of Column B there are relationships between depth-average values At the bottom there are relationships for values on the free surface and on rigid bottom In Column C there are period-average values They are functions of z or constants The constant values are checked by an asterisk

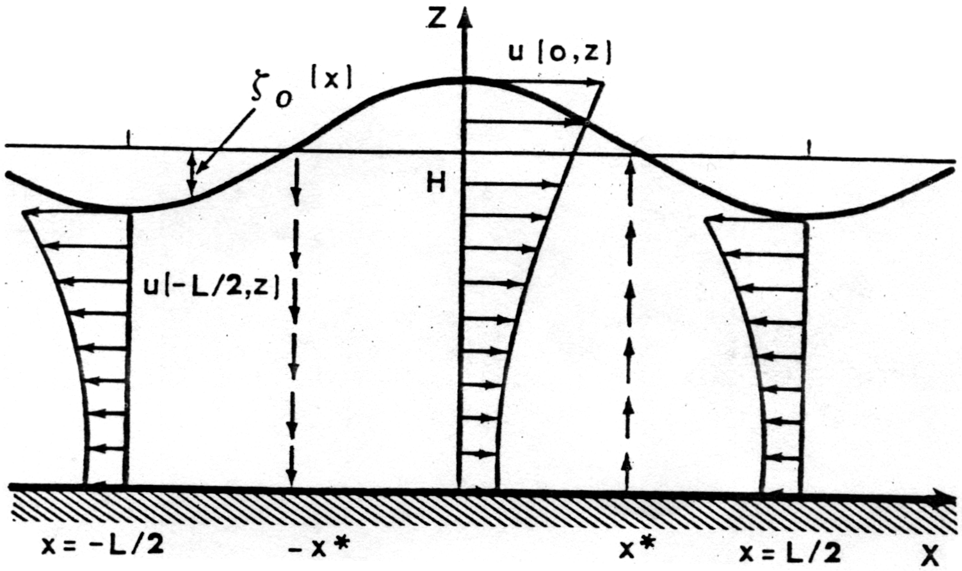

Figure 3 Notation and coordinates for the periodic wave

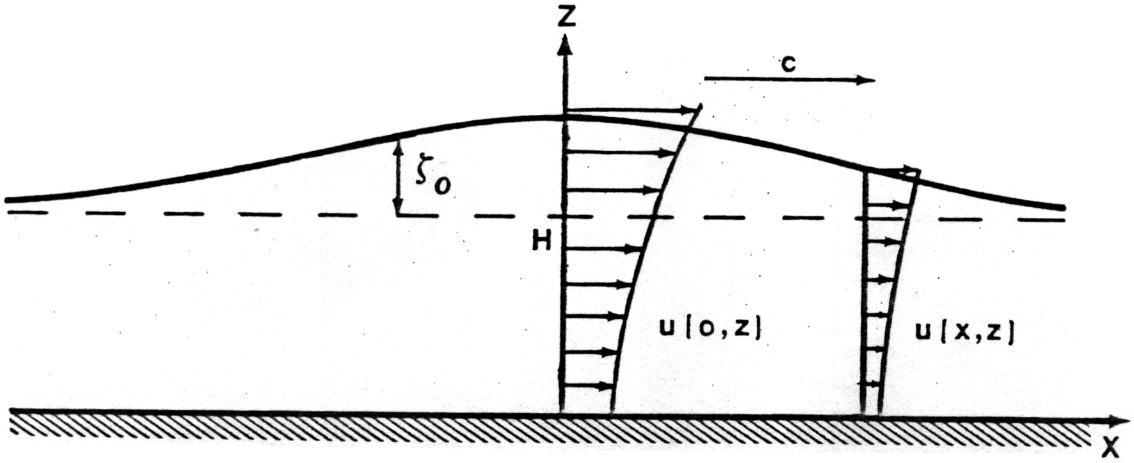

Figure 4 Notation and coordinates for the solitary wave

In Column A formulas are written in forms which are convenient for

ing

From Table 1 we can obtain that

$$c = \frac{\eta_o}{H}\sqrt{2\,(E_o - g\,\zeta_o)(1 + \zeta_{ox}^2)} \qquad (2.107)$$

Constant E_o is expressed through $\zeta_{ox}(x^*)$ as

$$E_o \quad \frac{c^2}{2}(1 + \zeta_{ox}^2\,(x^*\,)) \qquad (2.108)$$

In Table 2 there are relationships between functions at the points x L/2, 0 (at these points $\zeta_x = 0$ and $w = 0$) and at the point x^* (in these points $\zeta = 0$ and $\eta = 0$) We can express $\langle w^2\,(x^*)\rangle$ through ζ_o as

$$\frac{3}{2H} g\overline{\zeta_o^2} = E_o - \frac{c^2}{2} + \langle\frac{w^2(x^*)}{2}\rangle. \qquad (2.109)$$

All period-averaged quantities are expressed in terms of $\bar{u}$ and $\bar{u}_b$ The

quantities $\langle\bar{u}\rangle$ and $\bar{u}_b$ can be expressed in terms of ζ_o

We can obtain similar results for solitary waves (see Figure 4) There are differences only in boundary conditions with respect to x. For solitary waves we have

$$u,\ w\ \zeta\quad P \to 0 \qquad \text{at } x \to \pm\,\infty \qquad (2.110)$$

From these conditions we obtain that

Table 1 - Columr

1 $\eta = z + \zeta$

2 $w = (u - c)\,\zeta_x$

$$P = E_o - \frac{c^2}{2} + cu - \frac{u^2+w^2}{2}$$

4 $P_z = cw_x$

5 $u\,\eta_z = c\,\zeta_z$

6. $w\,\eta_z = -c\,\zeta_x$

7 $w\,\zeta_x = U - u$

$$(w\eta)_x = \frac{1}{c}[(\eta P)_z - E_o\eta_z] + \frac{c}{2} - u + \frac{1}{2}U$$

9. $u^2\,\eta_z = c^2\quad{}_z - cu$

10 $w^2\,\eta_z = c\,(u - U)$

11 $uw\eta_z = -c^2\,\zeta_x - cw$

12. $T = \frac{u^2 + w^2}{2}\,\eta_z = \frac{c}{2}(c\zeta_z - U)$

13 $u^3\eta_z = c^3\zeta_z - c^2u - cu^2$

Table 1, Column A (Continued)

14. $uw^2\eta_z = c^2(u - U) - cw^2$

$$P\,\eta = \frac{1}{2}\,cU + E_o\,\eta_z\quad\frac{c^2}{2}$$

16 $uP\eta_z = cE_o\zeta_z + c^2(\frac{U}{2} - u) + c\frac{u^2+w^2}{2}$

17 $F = u(T + P\eta_z) - c(E_o + \frac{c^2}{2})\zeta_z - c^2u$

18 $(T - \frac{1}{c}F)_x + (P\,\eta_x)_z = 0$

19 $U = c\zeta_z - (U\zeta)_z - \frac{c}{2}(\zeta^2)_{xx}$

20. U_z

21. $(U\,\eta)_z = c\,\zeta_z - \frac{c}{2}\,\eta^2{}_{xx}$

22 $(U\zeta)_z = -U + c\zeta_z - \frac{c}{2}\zeta^2{}_{xx}$

23. $g\zeta_o + \frac{1}{2}(u_s - c)^2(1 + \zeta_{ox}{}^2) = E_o$

24. $P = E_o - \frac{1}{2}(u - c)^2(1 + \zeta_x{}^2$

Table 1, Column B

1 $<\eta> = \frac{H}{2} + <\zeta>$

3 $<P> = E_o \quad \frac{c^2}{2} + c <u> - <\frac{u^2 + w^2}{2}>$

4 $cH <w>_x = g \zeta_o - P_b$

5 $<u\eta_-> = \quad \frac{\zeta_o}{H}$

6 $<w\eta_z> = - c <\zeta>_x$

7 $<w\zeta > = <U> - <u>$

8 $<w\eta>_x = \frac{g}{cH}\zeta_o\eta_o - \frac{E_o\eta_o}{cH} + \frac{c}{2} - <u> + \frac{<U>}{2}$

9 $<u^2 \eta_z> = c^2 \frac{\zeta_o}{H} - c <u>$

10 $<w^2 \eta_z> = c (<u> - <U>)$

11 $<uw\eta_z>_x = \frac{1}{H}P_b - g\frac{\zeta_o}{H} - c^2 <\zeta>_{xx}$

12 $<T> = \frac{c}{2} (c \frac{\zeta_o}{H} - <U>)$

13 $<u^3 \eta_z> = c^3 \frac{\zeta_o}{H} - c^2 <u> - c<u^2>$

Table 1, Column B (Conti

14 $<uw^2 \eta_z> = c^2 (<u> - <U>) - c<$

E_o

16 $<uP\eta_z> = cE_o \frac{\zeta_o}{H} + c^2 (\cdot <U>$

17 $<F> = c \frac{\zeta_o}{H} (E_o + \frac{c^2}{2}) - c^2 <u>$

18 $<u> = \frac{<U>}{2} + \frac{\zeta_o}{cH}(E_o - \frac{1}{2}g\zeta_o) + \frac{1}{c} < \frac{w^2(x^*)}{}$

19 $<U> = (c - U_s)\frac{\zeta_o}{H} - \frac{c}{2}<\zeta^2>_{xx}$

20 $U_s = c - \frac{cH}{\eta_o}(1 + \frac{1}{2}<\eta^2>_{xx})$

21 $(u_s - c)(1 + \zeta_{o_x}^2) = U_s - c$

22 $u_b = U_s + cH<\zeta>_{xx}$

23 $\frac{cH}{\eta_o}(1 + \frac{1}{2}<\eta^2>_{xx}) = \sqrt{2(E_o - g\zeta_o)(1+\zeta_{\cdot}}$

24 $P_b = E_o - \frac{1}{2} (u_b - c)^2$

Table 1 Column C

1 $\bar{\eta} = z \qquad \bar{\zeta} = 0$

2c

3 $\frac{1}{2}(\overline{u^2 + w^2}) = E_o - \frac{c^2}{2} + c\bar{u}$

4 $\bar{P} = 0$

5 $\overline{u\eta_z} = 0$

6 $\overline{w\eta_z} = -\frac{2c}{L} a$

8 $\zeta w_x = u - u_b$

9 $\overline{u^2\eta_z} = -c\bar{u}$

10 $w^2\eta_z = c(u - u_b)$

11 $\overline{uw\eta_z} = -\overline{u\zeta_x}$

12 $\bar{T} = -\frac{c}{2}\bar{u}_b$ *

13 $\overline{u^3\eta_z} = -c^2\bar{u} - c\overline{u^2}$

Table 1 Column C Continued

14 $\overline{uw^2\eta_z} = c^2(\bar{u} - \bar{u}_b) - c\overline{w^2}$

$\eta_z = -\frac{c}{2}$

16 $\overline{uP\eta_z} = c(E_o - \frac{c^2}{2} - \frac{c^2}{2}\bar{u}_b$

17. $\bar{F} = -c^2\bar{u}$

18. $\frac{3g}{2H}\zeta_o^2 = E_o - \frac{c^2}{2} + \frac{1}{2}\langle w^2(x^*)\rangle$

$\bar{U} = \bar{u}_b$ *

$c[u(0,z) - u(-L/2,z)] + (LP\eta_x - 2E_o a)_z = 0$

21 $cH[\langle u(0,z)\rangle - \langle u(-L/2,z)\rangle] = 2E_o a(H) - g[\zeta_o^2(0)$

22 $\bar{u}_b = c - \sqrt{2(E_o - g\zeta_o)(1 + \zeta_{o_x}^2)}$

23 $c = \frac{1}{H}\eta_o\sqrt{2(E_o - g\zeta_o)(1 + \zeta_{o_x}^2)}$

$\frac{1}{2}\overline{u_b^2} = E_o - \frac{c^2}{2} + c\bar{u}_b$

Table 2 Relations between properties of the periodic gravity wave at the crest $(x = 0)$ [or trough $(x = -L/2)$] and at the vertical $x = x^*$ where free surface elevation is equal to zero $(\zeta_o(x^*) = 0)$.

$x = -\frac{L}{2}$ or $x = 0$	$x = x^*$
$\zeta_x = 0$	$\zeta = 0,\ \zeta_{xx}$
$w = 0$	$u = 0$
$u_z = -c\,\zeta_{xx}$	$w = -c\,\zeta_x$
$P = E_o - \frac{1}{2}(c - u)^2$	$P = E_o \quad \frac{1}{2}\, c^2\,(1 + \zeta_x^{\ 2})$
$U = u$	$U = -c\,\zeta_x^{\ 2}$
$u_s - u_b = -c\,\langle\zeta_{xx}\rangle$	$\zeta_{ox}^{\ 2} = \frac{2\,E_o}{c^2} - 1$
$u_s = c - \sqrt{2\,(E_o - g\zeta_o)}$	$H\,\langle\zeta_{xx}\rangle = \frac{2\,E_o}{c^2} - 1$
	$\frac{1}{2}\,\langle\eta^2\rangle_{xx} = \frac{2\,E_o}{c^2} - 1$
	$P_b = E_o \quad \frac{c^2}{2}$
	$U_b = 0$
	$U_s = -Hc\,\langle\zeta_{xx}\rangle$

Table 3, Column A

1 $\eta = z + \zeta$

$w = (u - c)\,\zeta.$

$= cu - \frac{u^2+w^2}{2}$

4 $P_z = cw_x$

$u\,\eta_z = c\,\zeta_z$

6 $w\,\eta_z = -\,c\,\zeta_x$

7 $w\,\zeta_x = U - u$

$(w\eta)_x = \frac{1}{c} - \frac{c}{2}\zeta + \frac{c}{2} - u + \frac{1}{2}U$

$\eta_z = c^2\,\zeta_z - cu$

10 $w^2\,\eta_z = c\,(u - U)$

11. $uw\eta_z = -\,c^2\,\zeta_x - cw$

12 $T = \frac{u^2 + w^2}{2}\,\eta_z = \frac{c}{2}(c\zeta_z - U)$

13 $u^3\eta_z = c^3\zeta_z - c^2u - cu^2$

Table 3, Column A (Continued)

14 $uw^2\eta_z = c^2(u - U) - cw^2$

$= \frac{c}{2}$

16 $uP\eta_z = \frac{c^3}{2}\zeta_z + c^2(\frac{U}{2} - u) + c\frac{u^2+w^2}{2}$

17. $F = u(T + P\eta_z) = c^3\zeta_z - c^2u$

18 $(T - \frac{1}{c}F)_x + (P\,\eta_x)_z = 0$

19 $U = c\zeta_z - (U\zeta)_z - \frac{c}{2}(\zeta^2)_{xx}$

20 $U_z = -\,c\,\zeta_{xx}$

21. $(U\,\eta)_z = c\,\zeta_z - \frac{c}{2}\,\eta^2{}_{xx}$

22 $(U\zeta)_z = -U + c\zeta_z - \frac{c}{2}\zeta^2{}_{xx}$

23 $g\zeta_o + \frac{1}{2}(u_s - c)^2(1 + \zeta_{o_x}{}^2) = \frac{c^2}{2}$

24 $P = \frac{c^2}{2} - \frac{1}{2}(u - c)^2(1 + \zeta_x{}^2)$

Table 3 Column B

1 $\langle \eta \rangle = \frac{H}{2} + \langle \zeta \rangle$

3 $\langle P \rangle = c \langle u \rangle - \langle \frac{u^2 + w^2}{2} \rangle$

$cH \langle w \rangle_x = g \zeta_o - P_b$

$\langle u \eta_z \rangle = c \frac{\zeta_o}{H}$

6 $- c \langle \zeta \rangle_x$

7 $\langle w \zeta_{..} \rangle = \langle U \rangle - \langle u \rangle$

8 $\langle w \eta \rangle_x = \frac{g}{cH} \zeta_o \eta_o - \frac{c \zeta_o}{2H} - \langle u \rangle + \frac{\langle U \rangle}{2}$

9 $\langle u^2 \eta_z \rangle = c^2 \frac{\zeta_o}{H} - c \langle u \rangle$

10 $\langle w^2 \eta_z \rangle = c (\langle u \rangle - \langle U \rangle)$

11 $\langle u w \eta_z \rangle_x = \frac{1}{H} P_b - g \frac{\zeta_o}{H} - c^2 \langle \zeta \rangle_{xx}$

12 $\langle T \rangle = \frac{c}{2} (c \frac{\zeta_o}{H} - \langle U \rangle)$

13 $\langle u^3 \eta_z \rangle = c^3 \frac{\zeta_o}{H} - c^2 \langle u \rangle - c \langle u^2 \rangle$

Table 3 Column B (Continued)

14 $\langle u w^2 \eta_z \rangle = c^2 (\langle u \rangle - \langle U \rangle) - c \langle w^2 \rangle$

$+ \frac{c^2 \zeta_o}{}$

16 $\langle u P \eta_z \rangle = \frac{c^3 \zeta_o}{2H} + c^2 (\frac{\langle U \rangle}{2} - \langle u \rangle) + c \langle \frac{u^2 + w^2}{2} \rangle$

17 $\langle F \rangle = \frac{c^3 \zeta_o}{H} - c^2 \langle u \rangle$

18 $g \zeta_o^2 = c^2 \zeta_o + cH(\langle U \rangle - 2 \langle u \rangle)$

19 $\langle U \rangle = (c - U_s) \frac{o}{H} - \frac{c}{2} \langle \zeta^2 \rangle_{xx}$

20 $U_s = c - \frac{cH}{\eta_o} (1 + \frac{1}{2} \langle \eta^2 \rangle_{xx})$

21 $(u_s - c)(1 + \zeta_{o_x}) = U_s - c$

22 $u_b = U_s + cH \langle \zeta \rangle_{xx}$

23. $\frac{cH}{\eta_o} (1 + \frac{1}{2} \langle \eta^2 \rangle_{xx}) = \sqrt{2(c^2 - 2g\zeta_o)(1 + \zeta_{o_x}^2)}$

24 $P_b = c u_b - \frac{1}{2} u_b^2$

Table 3, Column C (Conti

14. $\overline{uw^2\eta_z}=c^2(\bar{u}-\bar{u}_b)-c\overline{w^2}$

$\overline{\text{p.}}$ 2

16 $\overline{uP\eta_z}=\frac{c^2}{2}\bar{u}_b+\frac{c^3}{2}\overline{\zeta_z}-gc\zeta_o+\frac{c^2}{2}\overline{u_b}$

17 $\bar{F}=c^3\overline{\zeta_z}-c^2\bar{u}$

18 $\bar{U}=\bar{u}_b$ *

19 $\frac{3g}{2}\zeta_o^2=(c^2-gH)\overline{\zeta_o}$

$\overline{(P\zeta_x)_z}=\frac{1}{2}u^2(0,z)\eta_z(0,z)$

$ga_o^2=c^2a_o-cH<u(0,z)>$

$ga_o=u_s(0)\ (c-\frac{u_s(0)}{2})$

$c=\frac{1}{H}\eta_o\sqrt{(c^2-2g\zeta_o)(1+\zeta_{o_x}^2)}$

$\frac{1}{2}u_b^2=c\overline{u_b}-g\overline{\zeta_o}$

Table 3 Column C

1 $\eta=z+\bar{\zeta}$

3 $g\bar{\zeta}_o=c\bar{u}-\frac{1}{2}(u^2+w^2)$

4. $\bar{P}=g\overline{\zeta_o}$

5 $\overline{u\eta_z}=c\overline{\zeta_z}$

6. $\overline{w\eta_z}=-ca$

7 $\overline{w\zeta_x}=\overline{u_b}-\bar{u}$

8 $\overline{(P\eta)_z}=\frac{c}{2}(c\overline{\zeta_z}+2\bar{u}-\bar{u}_b)$

9. $\overline{u^2\eta_z}=c^2\overline{\zeta_z}-c\bar{u}$

10. $\overline{w^2\eta_z}=c(\bar{u}-\bar{u}_b)$

11. $uw\eta_z=u\zeta_x$

12 $\bar{T}=\frac{c}{2}(c\overline{\zeta_z}-\bar{u}_b)$

13 $\overline{u^3\eta_z}=c^3\overline{\zeta_z}-c^2\bar{u}-c\overline{u^2}$

Table 4 Relationships between properties of the solitary wave at the crest ($x = 0$) and between average values.

$x = 0$	$\overline{<\ldots>}$
$w = 0, \quad \zeta_x = 0$	$\frac{1}{c}\overline{<F>} = 2\overline{<T>}$
$P = cu - \frac{u^2}{2}$	$H\overline{<u\eta_z>} = c\overline{\zeta_o}$
$U = u$	$g\overline{\zeta_o^2} = (\frac{c^2}{2} - gH)\overline{\zeta_o} + cH(\overline{<u>} - \frac{\bar{u}_b}{2})$
$2T = \frac{1}{c} F = u^2 \eta_z$	$\overline{<T>} + \frac{g}{2H}\overline{\zeta_o^2} = \frac{c^2}{H}\overline{\zeta_o} - c\overline{<u>}$
$P = g\zeta_o - c<w>_x$	$\overline{<T>} - \frac{g}{2H}\overline{\zeta_o^2} = c(\overline{<u>} - \bar{u}_b)$
$<T> = \frac{1}{2}(c^2 \frac{\zeta_o}{H} - c<u>)$	$\overline{<u^2 \eta_z>} = \frac{c^2}{H}\overline{\zeta_o} - c\overline{<u>}$
$u_s - u_b = -c<\zeta_{xx}>$	$\overline{<w^2 \eta_z>} = c(\overline{<u>} - \bar{u}_b)$
$g\zeta_o^2 = c^2 \zeta_o - cH<u>$	
$g\zeta_o = cu_s - \frac{1}{2} u_s^2$	
$u_s - <u> = -c<z\zeta_{xx}>$	

3 Approximated theories of nonlinear motions with a large horizontal scale

3.1 Governing equations

To consider some approximate theories we introduce the following dimensionless variable:

$(x,y) = L(x',y')$, $z = Hz'$, $t=Lt'/c$ $\varsigma=a\varsigma'$, $(u,v) = U(u\ \ v')$

$w = UHw/L$, $P = ag\rho_o(H)P'$ $h=H\cdot h'$

$N^2 = N_o{}^2N'^2$, $\Omega = \Omega_o\Omega'$, $\rho_o = \rho_o(H)\rho_o$

Here

L = typical lateral length scale,

H = typical vertical length scale,

a = typical amplitude displacement of isopycnal surfaces,

= typical phase velocity of waves or eddy translation velocity

= typical velocity of fluid particles

Dropping the primes we rewrite the problem (1.14) - (1.20) in terms of the rescaled dimensionless variables

$$D\overline{u} + \alpha\beta Dw\cdot\overline{\nabla}\varsigma + F\frac{1}{\rho_o}\overline{\nabla}P + R\,\overline{\Omega}\times\overline{u} = 0$$

$$\beta\,(1 + \alpha\varsigma_z)\,Dw + F\frac{1}{\rho_o}P_z + sN^2\varsigma = 0 \qquad 3.3)$$

$$Q\varsigma_{zt} + \overline{\nabla}\cdot[\overline{u}\ \ 1 + \alpha\varsigma_z) = 0 \qquad (3.4)$$

$$w = QD\varsigma \qquad (3.5)$$

he boundar: co ons

$$Q\zeta_t = \bar{u} \cdot \bar{\nabla} (h-\alpha\zeta) \quad \text{at } z = h*(x,y,t), \tag{3.6}$$

$$P = \zeta = \zeta_o \qquad \text{at } z = H \tag{3.7}$$

Here

$$U/C \quad \alpha = a/H, \ \beta = H^2/L^2, \ Q = \alpha/\varepsilon = ac/HU$$

$$F = ag/cU \quad Ro = \Omega_o L/c \quad s = N^2 aH/cU, \ F/Q = gH/c^2$$

$$D = \frac{\partial}{\partial t} + \varepsilon \left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}\right). \tag{3.8}$$

Assuming one or other dimensionless parameters small, we can reduce the system (3.2) - (3.5) to one equation with one unknown function.

3.2 Long Gently Sloping Waves

We will deal with long gently sloping waves, i.e we will neglect terms $\sim 0(\alpha\beta, \beta^2)$

$$\alpha\beta = aH/L^2, \ \beta^2 = H^4/L^4 \tag{3.9}$$

In this case, from system (3.2)-(3.7) we obtain

$$u_t + \varepsilon (uu_x + vu_y) + F \frac{1}{\rho_o} P_x - Ro\Omega v = 0, \tag{3.10}$$

$$v_t + \varepsilon (uv_x + vv_y) + F \frac{1}{\rho_o} P_y + Ro\Omega u = 0 \tag{3.11}$$

$$+ F \frac{1}{} P \quad \varsigma N^2 \zeta$$

$$Q\zeta_{zt} + \bar{\nabla} [\bar{u} (1 + \alpha\zeta_z) = 0, \tag{3.13}$$

$$w = Q\zeta_t + \varepsilon (u\zeta_x + v\zeta_y) \tag{3.14}$$

Henceforth, we will consider gravity waves and assume that $\alpha \sim \varepsilon$ and $\Omega = 1$ In this case equation (3.12) has the form

$$\beta Q\zeta_{tt} + F \frac{1}{\rho_o} P_{.} + sN^2\zeta = 0 \qquad (3 \quad 5)$$

When we consider large eddies and Rossby waves (Section 3.3) we will assume that $\varepsilon >> 1$ and $\Omega = 1 + \delta y \quad \delta << 1$

It is important that equation (3.12) is linear This makes it possible to reduce the system (3.10)-(3.14) to a system for functions independent of z

3.2.1 Homogeneous Fluid of Variable Depth

Let us put z = 0 at the undisturbed free surface

We seek a solution of the form

$$\zeta = A(x,y,t) + z\zeta_1 (x,y,t) + \beta\zeta_2(x,y,z,t) + 0(\beta^2), \tag{3.16}$$

$$\bar{u} = \bar{u}_1(x,y,t) + \beta\bar{u}_2(x,y,z,t) + 0(\beta^2) \tag{3.17}$$

From (3.13) we can obtain, using (3.6)-(3.7)

$$FP(x,y,z,t) = F\zeta_o - \beta \left[Q\zeta_{tt}(z + \frac{z^2}{2d}) + \frac{z^2}{2d} \bar{u}_t \cdot \bar{\nabla} d \right] \tag{3.18}$$

Substituting :his equatior to 3.10) 3.11 anc ntegrating wi respect to z from $-d^*(x,y,t)$ to 0 we obtain

$$Q\zeta_{ot} + \bar{\nabla} \quad \bar{U} \ (d + \alpha\zeta_o \qquad = 0 \tag{3.20}$$

$$\bar{U}_t + \varepsilon \ (U\bar{U}_x + V\bar{U}_y) + F \bar{\nabla}\zeta_o + \overline{Ro} \times \bar{U} +$$
$$+ \frac{-}{3} \beta \quad Qd^{1/2} \ \bar{\nabla} \ (\zeta_{ott} d^{1/2}) - \frac{1}{2} d^2 \bar{\nabla} \ (\ \bar{U} \quad \frac{\bar{\nabla} d}{d} \qquad = 0 \tag{3.21}$$

Here $\overline{Ro} = \{0,0,Ro\}$, $d^*(x,y,t) = 1 - h^*(x,y,t)$, $d(x,y) = 1 - h \ (x,y)$,

$$\bar{U} = \{U,V,0\} = \frac{1}{d^*} \int_{-d^*}^{o} \bar{u} dz$$

The system (3.20) - (3.21) is good even at the shoreline where the water depth $d^* \ (x,y(x,t),t)) = 0$.

In some particular cases it is possible to reduce system (3.20)-(3.21) to one equation (Odulo, 1978

3.2.2 Stratified fluid of constant depth

Following Ostrovsky (1978) we present the solution of the system (3.10), (3.11) (3.13), (3.15) in the form of an expansion of the eigenfunctions of the corresponding linear problem in the hydrostatic approximation

$$P = \sum_n \rho_o \Psi_n (z) P_n(x,y,t), \tag{3.22}$$

$$\bar{u} = \sum_n \Psi_1 (z)\ \bar{u}_{\cdot}\ (x,$$

where $\Psi_n (z)$ are the eigenfunctions of the problem

$$(\rho_o \Psi\ ) + c_n^{-2} \frac{Q}{F} s\ \rho_o N^2 \Psi = 0$$

$$\Psi = 0 \qquad \text{at} \quad z = 0 \quad H \tag{3.23}$$

Here n is the number of the mode. The orthogonality conditions have the form

$$<\rho_o N^2 \Psi_k \Psi_m> = <\rho_o \Psi_k\ \Psi_m\ > = 0 \quad k \neq m \tag{3.24}$$

where $< \quad > \equiv \frac{1}{H} \int_o^H \cdot \ dz$

Substituting (3.22) into (3.10), (3.11), (3.13), (3.14) (3.15) and performing the usual orthogonilization procedure, we obtain the single mode approximation (dropping the index)

$$\bar{u}_t + \varepsilon S_n (u\bar{u}_x + v\bar{u}_y) + \bar{\nabla} (c_n^2 \frac{F}{Qs} \zeta + D_n Q\beta\zeta_{tt}) + \bar{R}o \times \bar{u} = 0$$

$$Q\zeta_t + \bar{\nabla} [\bar{u} (1 + \alpha S_n \zeta) = 0 \tag{3.25}$$

$$<\rho_o \Psi_{nz}{}^3> \qquad <\rho \Psi_n{}^2>$$

$$_o \Psi_{nz} \qquad <\rho_o \Psi_{nz}$$

It is easy to see, that in the case N^2 = const nonlinear terms in (3.25) are small, since

$$S_n \quad \frac{\rho(0) - \rho(H)}{\rho(H)} \ll 1$$

If, in equation (3.21), we put depth $d = 1$ and, in equation (3.25), $S_n = 1$, $D_n = 1/3$, $c_n{}^2 = Qs$, we will get the same equations

3.2.3 Two layer fluid of variable depth

Consider a two-layer incompressible fluid (upper layer at $0 < z <$ lower layer at $-d(x) < z < 0$) Using (3.20) - (3.21) we can see that the motion is described by equations

$$Q\zeta_{ot} + [U_1 \; (-1 + \alpha\zeta_o)]_x = 0 \tag{3.27}$$

$$U_{1t} + \varepsilon U_1 U_{1x} + F \, P_{1x} - \frac{1}{3} \beta Q\zeta_{ottx} = 0 \tag{3.28}$$

$$Q\zeta_{ot} + [U_2 \; (d + \alpha\zeta_o)]_x = 0 \tag{3.29}$$

$$U_{2t} + \varepsilon U_2 U_{2x} + FP_{2x} + \frac{1}{3} \beta \left[\qquad \frac{d^{1/2}}{2} \left(\frac{U_{2t} d_x}{d} \right)_x \right] = 0 \tag{3.30}$$

$$sP_2 - P_1 = \zeta_o \; (s-1) \tag{3.31}$$

whe dz $\int_d^o u_o dz$ $s = \rho_1/\rho_2$

We introduce a new function

$$U = s\,U_2 - U_1 \;=\; U_2\left[s + \frac{d + \alpha\zeta_0}{-\alpha\zeta_0} \right] = \frac{s + d + \alpha\zeta_0(s-1)}{-\alpha\zeta_0} \tag{3.32}$$

From (3.27), (3.29) we obtain

$$U_1 = -\frac{-d + \alpha\zeta_o}{1 - \alpha\zeta_o}\,U_2 \tag{3.33}$$

and using (3.32) we have

$$U = U_2\,(s + d)\,\frac{1 - \alpha a^2\zeta_o}{1 - \alpha\zeta_o} \tag{3.34}$$

here $$a^2 = \frac{s - 1}{s + d} \tag{3.35}$$

From (3.29) and (3.34) we have

$$Q\zeta_{ot} + \left[\frac{U}{}\;\frac{(d + \alpha\zeta_o)\,(1 - \alpha\zeta_o)}{1 - \alpha\zeta_o\,a^2}\right]_x = 0 \tag{3.36}$$

From (3.28), (3.30), (3.33) and (3.34) we obtain

$$U_t + \frac{\varepsilon}{2}\left[\frac{U^2}{s+d}\;\frac{b - 2\,\alpha\zeta_o + \alpha^2 a^2\zeta_o^2}{(1 - \alpha a^2\zeta_o)^2}\right]_x + F\,(s - 1)\,\zeta_{ox} +$$

$$+ \frac{1}{3}\,\beta Q\zeta_{ottx} + \frac{s}{3}\,\beta\left[Qd^{1/2}\,(\zeta_{ott}\,d^{1/2})_x - \frac{d^2}{2}\;\frac{U_t d_x}{d(s + d)}\right)_x = 0, \tag{3.37}$$

where $$b = \frac{s - d^2}{s + d}$$

function

$$Q\zeta_o = -\Psi_x \tag{3.38}$$

$$\frac{U}{} = \frac{1 + \varepsilon a^2\Psi_x}{(d-\varepsilon\Psi_x)(1 + \varepsilon\Psi_x)} \tag{3.39}$$

Then from (3.37) we obtain

$$\left[\frac{(1 + \varepsilon a^2\Psi_x)\Psi t}{(d-\varepsilon\Psi_x)(1+\varepsilon\Psi_x)}\right]_t + \frac{\varepsilon}{2(s+d)}\left[\Psi_t^2 \frac{(s + d)(b + 2\varepsilon\Psi_x + \varepsilon^2 a^2\Psi_x^2)}{(d-\varepsilon\Psi_x)^2 (1 + \varepsilon\Psi_x)^2}\right]_x$$

$$- c_o^2\Psi_{xx} \quad \frac{s\beta}{3(s+d)}\left[d^{1/2} (\Psi_{xtt}d^{1/2})_x + \frac{d^2}{2}(\Psi_{tt}\frac{d_x}{d^2})_x\right. = 0, \tag{3.40}$$

where $c_o^2 = a^2gH/c$

When $b = 0$, it is easy to show that all quadratic terms in (3.40) vanish, but other nonlinear terms do not The evolution of weakly-nonlinear two-layer flow over topography was considered by Helerich & Melville (1984, 1986, 1987).

3.3 Large scale eddies on "β-plane"

Now we want to consider Rossby waves or eddies and introduce a Rossby number

U

LΩ(

We assume also, that

$$F \sim Ro \quad \Omega = 1 + \delta y \qquad \frac{L}{r_o} << 1 \qquad 3$$

where r_o is the Earth radius

We consider motions with large lateral scale and assume

$$< 1, \ \varepsilon \geq 1 \tag{3.42}$$

We rewrite equations (3.2) - (3.4) in the form

$$-\Omega u = \frac{F}{Ro} \left(\frac{P}{\rho_o}\right)_y + \frac{\mu}{\varepsilon} Dv + \mu Q \beta D w \zeta_y, \tag{3.43}$$

$$\Omega v = \frac{F}{Ro} \left(\frac{P}{\rho_o}\right)_x + \frac{\mu}{\varepsilon} Du + \mu Q \beta D w \zeta_x, \tag{3.44}$$

$$-sN^2 \zeta = \frac{F}{Ro} P_z + \beta Q \ (1 + \alpha \zeta_z) \ DD\zeta, \tag{3.45}$$

And equation (1.35) has the form

$$\frac{\mu}{\varepsilon} (1 + \alpha \zeta_z) \omega_3 + (1 + \alpha \zeta_z) \ \delta \rho_o v - (\mu \omega_3 + \rho_o \Omega) \ Q D \zeta_z = 0. \tag{3.46}$$

We can see from (3.46) that

$$Q \leq \max \ (\mu \quad \delta) \tag{3.47}$$

Let

$$\delta \sim Q << 1 \tag{3.48}$$

If we exclude terms of order

$$\alpha\beta \quad \mu^2 \quad \delta\mu \quad \alpha\mu \tag{3.49}$$

obtain from

$$\frac{\mu}{\varepsilon} \Delta P_t + \mu \frac{F}{Ro\rho_o} J(P, \Delta P) + \left[1 - \alpha \left(\frac{FP_z}{sN^2\rho_o}\right)_z \right] \delta P_x + \tag{3.50}$$

$$+ Q\ (1 + \delta y) \quad \left(\frac{FP_z}{sN^2\rho_o}\right)_{zt} + \alpha J\left(P, \left(\frac{FP_z}{sN^2\rho_o}\right)_z\right) = 0$$

This equation is known (Pedlosky, 1987) but we obtain it here using fewer assumptions, only (3.42) and (3.48)

4. CONCLUSIONS

Whe we conside: near motion in a fluid with free transfer the boundary conditions to the undisturbed level and solve the problem in a fixed region. In the nonlinear case it is difficult to solve the problem in an unknown (time variable) region. The motivation for introducing the new functions was to obtain the problem in a fixed region, without complicating the equations. The new functions introduced in §1.2 satisfy these conditions In the case of a flat bottom, the domain of definition of new functions is a layer of constant thickness We considered the case of variable-depth fluid only for long gently sloping waves. For these only the undisturbed water depth appears in the final equations (3.20) - (3.21) in §3.2.1, and (3.40) in §3.2.3.

It is a notable advantage that derivatives with respect to z do not appear in horizontal projections of the momentum equations (1.14) It is therefore possible to exclude z-dependence in equations describing motion of a large horizontal scale (Chapter 3) and to integrate the corresponding equations for plane steady motion. A second important advantage of the approach described here is that derivatives with respect to z are present only in two terms, P_z and ζ_z, and there are not derivatives of the velocity components with respect to z in the system (1.14)-(1.17)

For the equations structured in this way it is possible to write the mechanical energy equation in a divergence form (1.30) and to obtain the impulse and energy conservation laws in the forms (1.28) (1.29) and (1.31) accordingly. Moreover we can separate the kinetic energy

$$= \frac{1}{2} \rho_o \eta_z \quad 2 \quad 2 \quad 2$$

the baroclinic potential energy

$$i = \frac{1}{2} <\rho_o N^2 \zeta^2 \tag{4 2}$$

and the barotropic potential energy

$$\Pi^s = \frac{1}{2} g \rho_{os} \zeta_o^2 \tag{4.3}$$

of a fluid column (§1.3) It is possible to write the Hamiltonian in the obvious form (1.55) (§1.6)

It is clear that if we exclude nonlinear terms the expressions (1.9) become identities and the definition of the function $\zeta(x,y,z,t)$ reduces to

$$w(x,y,z,t) = \zeta_t(x,y,z,t \tag{4.4}$$

Hence this theory is important only for nonlinear problems

It is also clear that if vertical component of velocity and a vertical displacement of the isopycnal surfaces are both zero, expressions (1.7) (1.9) are identical and equations (1.14) - (1.17) will be exactly the same as equations (1.1) - (1.3)

To represent the vorticity we introduce a new function $\overline{V}$ (see 1.32) For irrotational motion of homogeneous nonrotating fluid, the momentum equation for function $\overline{V}$ has a simple form

$$\overline{v}_t + \overline{\nabla}_3(\overline{u} \quad \overline{V} - T + P) = 0 \tag{4.5}$$

The function $\bar{V}$ is also useful when we consider the integr relations i1 Chapter 2 (see formulas (2.43 (2.46 and 2.102 (2.105) and ables 1-4

It s interesting to compare the vorticity equations for dimensional motions of homogeneous nonrotating fluid in a vertical plane (x,z)

$$\omega_{2_t} + (u\omega_2)_x = 0 \tag{4.6}$$

and in a horizontal plane (x,y

$$\omega_{3_t} + (u\omega_3)_x + (v\omega_3)_y = 0 \tag{4.7}$$

Note that equation (4.7) is identically the same for new and old functions.

When homogeneous inviscid fluid moves under conservative forces we know that a vortex line consists always of the same fluid particles, and vortex filaments must be either closed or terminated at the boundaries But we exclude motions having closed fluid filaments in a vertical plane, therefore for motion in a vertical plane we have that

$$\frac{\partial}{\partial t} \int_{x_2}^{x_1} \omega_2 dx = u\omega_2 \Big|_{x_2}^{x_1}$$

We need the function $\bar{V}$ also for the Clebsch representation of the velocity to give the variational principle (§1.5).

In Chapter 2 we considered two-dimensional motion of nonrotating fluid showed that the problem can be reduce to one equ tior and integral relations in the general case of rotational motion of a stratified fluid Thereafter making more and more assumptions (steady motion, homogeneous fluid, irrotational motion) we obtained simpler equations and more relations for integral properties of the motion Since the domain of definition of the functions is a strip of constant thickness, we have

$$\;_x> = < \quad x \qquad t> = < \quad >_t \tag{4.9}$$

and

$$\bar{z} = \overline{\cdots z}, \quad \cdot t = \quad t \tag{4.10}$$

This means we can obtain integral relations from corresponding equations very easily

In Tables 1 and 3 there are the equations in Column A, depth-average values in Column B, and half-period-average values in Column C

Analogously we can obtain integral relations for averaged higher order values (see (2.68) - (2.75) and (2.81) - (2.87))

It is easy to compare these integral relations with previous results (Lonquet-Higgins, 1974, 1975, 1980, 1984, 1988; Yu and Wu, 1987) if we use the relation

$$\int_0^{H+\zeta_0} f(x,y,z,t)dz = \int_0^H f(x,y,z,t)\eta_z dz \tag{4.11}$$

for any function f For example we have

$$u^2+w^2)dz = \int_o^H (u^2+w^2)\eta_\sim dz$$

In Chapter 3 we considered motions with large horizontal scale.

system (3.20 (3.21) and the equation (3.40) have no singularities at the shoreline and can be used to calculate the solution to problems concerning nonbreaking waves on a slope. Equation (3.40) can also be useful also in studying the transformation of nonlinear internal wave propogation through the point where $\rho_1 H_1^2 = \rho_2 H_2^2$ ($\rho_1, \rho_2, H_1, H_2(x)$ are density and depth in upper and lower layers) At this point the coefficient of quadratic terms changes sign.

introduction of the functions (1.9) is a very useful tool in studying nonlinear motions of an inviscid stratified fluid.

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